Young Geometric Group Theory 1
January 9-13, Bedlewo

http://www.impan.pl/BC/Arch/2012/Young.html

Young Geometric Group Theory 2
February 3-7, Haifa

www.math.technion.ac.il/cms/decade_2011-2020/year_2012-2013/

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Lecture 4: Special cube complexes

Separability

G - finitely generated group, $H < G$

H is said to be <u>separable</u> if for any $g \in G - H$ there exists a finite index subgroup K such that
$H < K$ and $g \notin K$.

<u>Exercise.</u> H is separable $\Leftrightarrow$ for every $g \in G - H$, there exists a finite quotient f:

$G \to F$, in which $f(g) \notin f(H)$.

G is <u>residually finite</u> if the trivial group is separable.

$G = \pi_1(K) \Rightarrow$ can "unwrap" loops (and not others)

The profinite topology on G is the topology defined by taking cosets of finite index subgroups as basic open sets.

Exercise. G is residually finite $\Leftrightarrow$ G is Hausdorff.

Exercise. H is separable $\Leftrightarrow$ H is closed.

Exercise. If G is residually finite and there H is virtual retract of G, then H is separable.

Stallings proof of Marshall Hall's theorem

Theorem. Every finitely generated subgroup of a finitely generated free group is separable.

Proof. Completion and retract....

Special cube complexes (Haglund-Wise)

Definition. A NPC cube complex is called special if the it satisfy:

1. Every hyperplane is embedded.

2. Every hyperplane is 2-sided.

3. No hyperplane self-osculates.

4. No two hyperplanes interosculate .

Recall: A Salvetti complex for a RAAG.

Lemma. A cube complex is special $\Leftrightarrow$ it admits a locally isometric immersion into a Salvetti complex.

Proof:

$\Leftarrow$ Definitive properties are preserved by pullbacks.

$\Rightarrow$ Let Γ = the incidence graph of the hyperplanes

Build map $X \to X_\Gamma$ in the natural way.

Recall: Since special complexes come with locally isometric maps to NPC complexes, they are NPC.

Completion and Retraction

A - compact cube complex, R - Salvetti complex

$f: A \to R$, a locally isometric immersion. We describe $C(A,R)$, the canonical completion of A.

$C^1(A,R)$ - define as before for graphs.

$C^n(A,R)$ - Fill in cubes, inductively, to obtain $C(A,R)$, a covering space of R

The retraction $C^1(A,R) \to A^1$ extends to a retraction $C(A,R) \to A$

Completing a map to a special complex

Fiber product: a cell for every pair of cells, one in C(A,R) and one in B, that map to the same cell in R.

$$
\begin{tikzcd}
 & {} \arrow[d] & \\
 & B \otimes_R C(A,R) \arrow[r] \arrow[d] & C(A,R) \arrow[d] \\
A \arrow[r] & B \arrow[r] & R \\
 & \text{special} \arrow[u] &
\end{tikzcd}
$$

Example: quasiconvex subgroups

X – special cube complex

G – $\pi_1(X)$, Gromov hyperbolic

$H < G$, quasiconvex subgroup

We show that H is a virtual retract $\Rightarrow$ separable.

Cores (Haglund)

CAT(0)

$\downarrow$

v - a vertex of Y = universal cover of X

$$\mathrm{Hull}(Hv) \equiv \bigcap_{Hv \in \mathfrak{h}} \mathfrak{h}$$

Proposition. There exists $R>0$ such that
$\mathrm{Hull}(Hv) \subset N_R(Hv)$.

<u>Proof.</u> If point is far away from Hv, there exists a hyperplane separating it from Hv...

Separability argument

- $A = \mathrm{Hull}(Hv)/H$ is compact and immerses into X

- $\pi_1(A) = H$

- Form canonical completion $C(A,X)$

- We then have a retraction $C(A,X) \to A$

- Conclusion H is a virtual retract so is separable.

Agol's theorem

Theorem. X a Gromov hyperbolic CAT(0) cube complex, $G \circlearrowright X$ properly and cocompactly, then there exists a finite cover of X/G which is a special cube complex.

Corollary. M a closed hyperbolic 3-manifold, then every quasi-convex π_1 -injective surface in M lifts to an embedding in a finite cover.

$\Downarrow$

Virtual Haken conjecture

Corollary. $C'(1/6)$ - groups are virtually special.
In particular, they are linear!