Lecture 3: Rank Rigidity

Definition. A hyperbolic isometry in a CAT(0) space is called rank 1 if an (any) axis for the isometry does not bound a half-flat (isometric embedding of the Euclidean upper half plane).

Example. G a hyperbolic group acting properly and cocompactly on CAT(0) space then every element is rank 1

Rank 1 elements behave like hyperbolic isometries in hyperbolic space.

Exercise (nonexample). A group acting properly and cocompactly on a product of two trees.

Rank Rigidity Conjecture (Ballmann-Buyalo). G acts properly and cocompactly on a (nice) CAT(0) space X then either

1. G contains rank 1 elements
2. X is a product
3. X is a higher rank symmetric space or a Euclidean building

Rank Rigidity For CAT(0) cube complexes (Caprace-S). G acts properly, cocompactly, on a CAT(0) cube complex X then either

1. G contains rank 1 elements
2. X contains a convex invariant subcomplex which splits as a product

Essentiality

An action of G on X is called essential if every halfspace contains points arbitrarily far away from its bounding hyperplane

Lemma. $G \circlearrowright X$ cocompactly then there exists an equivariant convex subcomplex on which G acts essentially.

Proof.

- <u>essential</u>, <u>half-essential</u> and <u>inessential</u> hyperplanes

- equivariantly collapse half-essential hyperplanes from the outside in to obtain a subcomplex Y with no half-essential hyperplanes

- every essential hyperplane crosses every inessential hyperplane

$$\Downarrow$$

- $Y = Z \times C$ where C is compact and all hyperplanes in Z are essential

Assume from now on that G acts on X essentially.
In fact, for simplicity, we will make the following

$\boxed{\text{Standing Assumption: extendible geodesics.}}$

<u>Recall:</u> X decomposes as a product if and only if the hyperplanes of X decompose as a disjoint union:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 \cup \hat{\mathcal{H}}_2$$

Where each hyperplane in $\hat{\mathcal{H}}_1$ crosses each hyperplane in $\hat{\mathcal{H}}_2$.

We will now assume this is not the case.

Skewering

$g \in \mathrm{Aut}(X)$ is said to skewer a halfspace $\mathcal{h}$ if $g\mathcal{h} \subset \mathcal{h}$

Exercise:

1. g skewers $\mathcal{h} \Rightarrow$ any axis of g crosses $\hat{\mathcal{h}}$.

2. the axis of g crosses $\hat{\mathcal{h}} \Rightarrow g^n$ skewers h, for some n.

Note that g skewers $\mathcal{h} \Leftrightarrow g^{-1}$ skewers $\mathcal{h}^*$. We will sometimes say that g skewers the hyperplane $\hat{\mathcal{h}}$.

Single Skewering

Lemma. $G \circlearrowright X$ cocompactly then
every hyperplane is skewered by some element of G.

Proof.

- $\hat{h}$ a hyperplane
- Consider orbit quotient $X(G\hat{h})$
- There exists a quotient map $p:X \to X(G\hat{h})$

- If diam($X(G\hat{h})$) is big enough there exist 3 disjoint hyperplanes along a geodesic in $X(G\hat{h})$ $\Rightarrow$ skewering

- Otherwise, $X(G\hat{h})$ is bounded

- There exists a fixed point v for G in $X(G\hat{h})$

- $p^{-1}(v) \subset X$ is a fixed region bounded by hyperplanes $\Rightarrow\Leftarrow$

Flipping

$g \in \mathrm{Aut}(X)$ is said to flip a halfspace $\mathcal{h}$ if

$$g\mathcal{h} \subset \mathcal{h}^*$$

We say that h is unflippable if there does not exist g flipping h.

Flipping Lemma: Suppose that $G \circlearrowright X$ properly, cocompactly essentially with an unflippable hyperplane. Then X decomposes as a product:

$$X = Y \times \mathbb{R}$$

Proof.

- h - an unflippable halfspace

Goal: Show that for every hyperplane $\hat{k}$ disjoint from $\hat{h}$, the hyperplanes crossing $\hat{k}$ are the same as the hyperplanes crossing $\hat{h}$

- Step 1. If $\hat{k} \subset h^*$, then every hyperplane crossing $\hat{k}$ crosses $\hat{h}$.

- Step 1 gives an isometric embedding $\hat{k} \to \hat{h}$

Exercise (Endometry Lemma). If M is a proper cocompact metric space and f: M → M is an isometry, then f is onto.

- Step 1 together with the Endometry Lemma gives:

- Step 2. For every hyperplane translate $g\hat{h} \subset h^*$,
  a hyperplane crosses $\hat{h} \Leftrightarrow$ it crosses $g\hat{h}$.

- <u>Step 3.</u> If $\hat{k} \subset h^*$, then a hyperplane crosses $\hat{h} \Leftrightarrow$ it crosses $\hat{k}$

- <u>Step 4.</u> If $\hat{k} \subset h$, then a hyperplane crosses $\hat{h} \Leftrightarrow$ it crosses $\hat{k}$

So we have accomplished our goal: the hyperplanes that cross $\hat{h}$ are the same as hyperplanes that cross a hyperplane disjoint from $\hat{h}$.

Let $\hat{\mathcal{H}}_{\parallel}$ = {hyperplanes disjoint from $\hat{h}$}

$\hat{\mathcal{H}}_{\perp}$ = {hyperplanes intersecting $\hat{h}$}

Conclusion: We can now write $\mathcal{H}$ as the disjoint union

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\parallel} \cup \hat{\mathcal{H}}_{\perp}$$

and every hyperplane in $\hat{\mathcal{H}}_{\parallel}$ intersects every hyperplane in $\hat{\mathcal{H}}_{\perp}$

- $\Rightarrow$ X decomposes as a product

$$X = X_{\parallel} \times X_{\perp}$$

- $X_{\parallel}$ is a tree.

- There is a copy T of $X_{\parallel}$ which appears in X as a maximal intersection of hyperplanes meeting $\hat{h}$.

- Stab(T) < G acts on T properly and cocompactly and has an unflippable halfspace.

- So $T = X_{\parallel}$ is a line.

Double Skewering

Lemma. $G \circlearrowright X$ cocompactly and essentially then every for every pair of nested halfspaces $h \subset k$, there is some element of $g \in G$ such that $gk \subset h$.

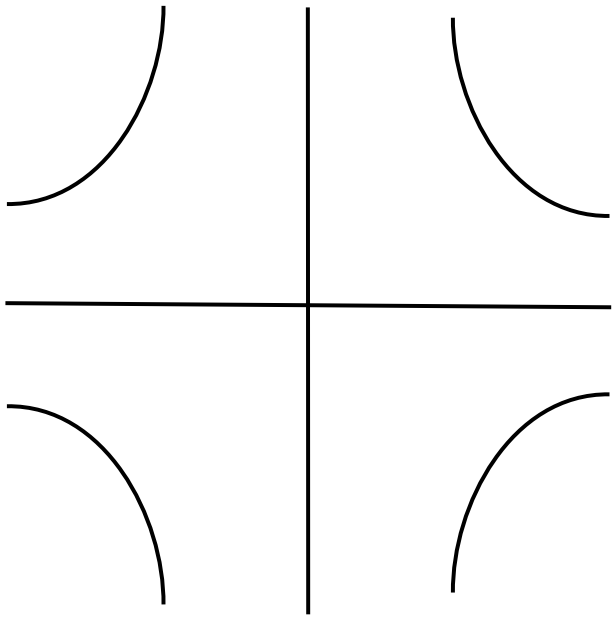
Proof.

- If either h or k are unflippable, then by the Flipping Lemma, we have that $\hat{h}$ and $\hat{k}$ are "parallel" and some power of any element skewering one skewers the other.

- If both h and k are flippable flip twice

Sectors contain hyperplanes

Lemma. If $\hat{h}$ and $\hat{k}$ are intersecting hyperplanes, then all four complementary regions (sectors) of
$\hat{h} \cup \hat{k}$ contain a hyperplane.



Proof.

- Step 1. Assume there exists some hyperplane disjoint from $\hat{h}$ and $\hat{k}$.

- Step2. Assume that every hyperplane either crosses $\hat{h}$ or crosses $\hat{k}$

$\hat{\mathcal{H}}$ = { hyps disjoint from $\hat{h}$ }

$\hat{\mathcal{K}}$ = { hyperplanes disjoint from $\hat{k}$ }

$\hat{\mathcal{H}}'$ = { hyperplanes disjoint from some hyp $\in \hat{\mathcal{H}}$}

$\hat{\mathcal{K}}'$ = { hyperplanes disjoint from some hyp $\in \hat{\mathcal{K}}$}

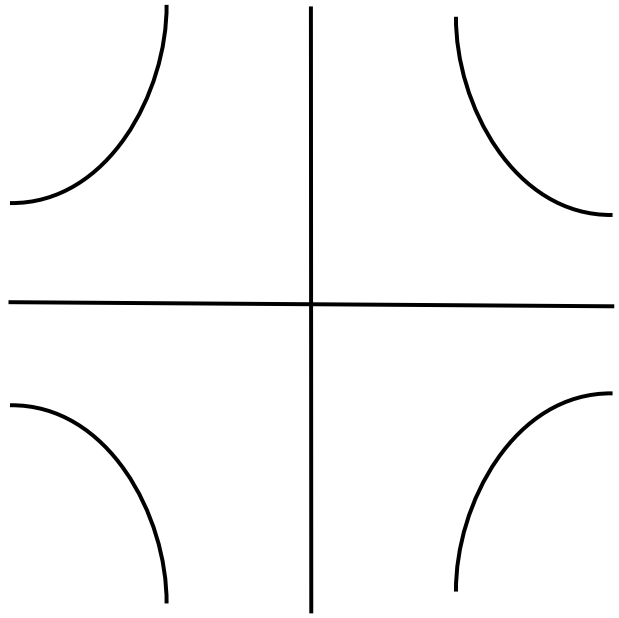
$\hat{\mathcal{R}} = \hat{\mathcal{H}} - (\hat{\mathcal{H}}' - \hat{\mathcal{K}}')$

- Show that $\hat{\mathcal{H}}' \cup \hat{\mathcal{K}}' \cup \hat{\mathcal{R}}$ is a transverse decomposition of $\hat{\mathcal{H}}$.

Application: Tits Alternative

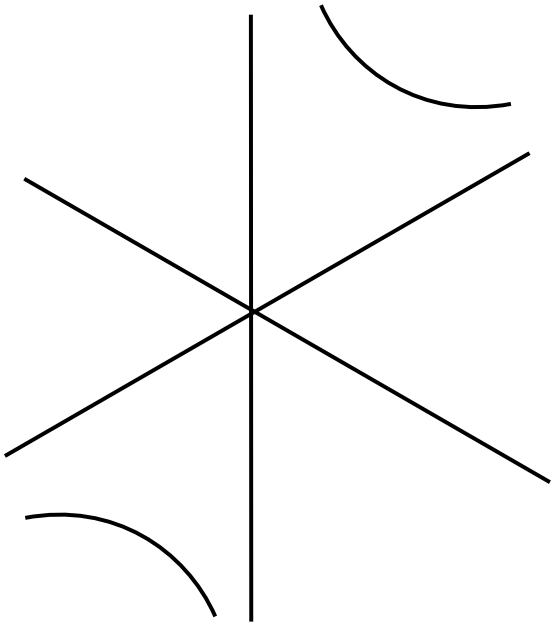
Corollary. X irreducible and not a line $\Rightarrow$ G has a free subgroup of rank 2.

Proof. Ping pong.



1. Consider a maximal collection of intersecting hyperplanes having diagonally opposite sectors containing hyperplanes.

2. Find g skewering those hyps



4. Find a hyp crossing all of these.

3. Axis for g bounds a half-flat.

5. Use Sector Lemma to contradict maximality