Definition. A hyperbolic isometry in a CAT(0) space is called rank 1 if an (any) axis for the isometry does not bound a half-flat (isometric embedding of the Euclidean upper half plane).

Example. A group acting properly and cocompactly on a product of two trees.

Rank 1 elements behave like hyperbolic isometries in hyperbolic space.

Exercise (nonexample). A group acting properly and cocompactly on a CAT(0) space.

Example. A hyperbolic group acting properly and cocompactly on CAT(0) space.

Definition. A hyperbolic isometry in a CAT(0) space is called rank 1 if an (any) axis for the isometry does not bound a half-flat (isometric embedding of the Euclidean upper half plane).
Rank Rigidity Conjecture (Ballmann-Buyalo). G acts properly and cocompactly on a (nice) CAT(0) space X then either
1. G contains rank 1 elements
2. X is a product
3. X is a higher rank symmetric space or a Euclidean building

Rank Rigidity For CAT(0) cube complexes (Caprace-S). G acts properly, cocompactly, on a CAT(0) cube complex X then either
1. G contains rank 1 elements
2. X contains a convex invariant subcomplex which splits as a product

Essentiality

An action of G on X is called essential if every halfspace contains points arbitrarily far away from its bounding hyperplane.

Lemma. G ∪ X cocompactly then there exists an equivariant convex subcomplex on which G acts essentially.
Proof.

- essential, half-essential and inessential hyperplanes

\[ Y = Z \times C \text{ where } C \text{ is compact and all hyperplanes in } Z \text{ are essential} \]

\[ \uparrow \]

every essential hyperplane crosses every inessential hyperplane

obtain a subcomplex \( Y \) with no half-essential hyperplanes

equivariantly collapse half-essential hyperplanes from the outside in to essential, half-essential and inessential hyperplanes

Proof
Assume from now on that $G$ acts on $X$ essentially. In fact, for simplicity, we will make the following

Recall: $X$ decomposes as a product if and only if the hyperplanes of $X$ decompose as a disjoint union:

$$H = H_1 \cup H_2$$

Where each hyperplane in $H_1$ crosses each hyperplane in $H_2$.

We will now assume this is not the case.

Standing Assumption: extendible geodesics.
Skewering $g \in \text{Aut}(X)$ is said to skewer a halfspace $h$ if $g^{-1}$ skewers $h$. We will sometimes say that $g$ skewers the hyperplane $h$. Note that $g$ skewers $h$ if and only if $g^{-1}$ skewers $h$. We will sometimes say that $g$ skewers the hyperplane $h$. 

**Exercise:**
1. $g$ skewers $h$ if and only if any axis of $g$ crosses $h$.
2. The axis of $g$ crosses $h$ if and only if $g^n$ skewers $h$, for some $n$. 

$g \in \text{Aut}(X)$ is said to skewer a halfspace $h$ if $g h \subset h$. 

Skewering
There exists a quotient map \( p : X \rightarrow X(\mathcal{G}_h) \).

Consider orbit quotient \( X(\mathcal{G}_h) \).

Every hyperplane is skewered by some element of \( G \).

**Proof**

Single Skewering
If diam$(X(G_h))$ is big enough there exist 3 disjoint hyperplanes along a geodesic in $X(G_h) = \text{skewering}

\Rightarrow \Leftarrow p^{-1}(v) \subset X$ is a fixed region bounded by hyperplanes

There exists a fixed point $v$ for $G$ in $X(G_h)$

Otherwise, $X(G_h)$ is bounded
Flipping $g \in \text{Aut}(X)$ is said to flip a halfspace $h$ if $g h \subset h^*$. We say that $h$ is unflippable if there does not exist $g$ flipping $h$.

**Flipping Lemma:** Suppose that $G \acts X$ properly, cocompactly essentially with an unflippable hyperplane. Then $X$ decomposes as a product:

$$X = Y \times \mathbb{R}$$
Proof.

Step 1 gives an isometric embedding $h \rightarrow k$.

Goal: Show that for every hyperplane $k$ disjoint from $h$, the hyperplanes crossing $k$ are the same as the hyperplanes crossing $h$.

Step 1. If $k \subset h^*$, then every hyperplane crossing $k$ crosses $h$.

Goal: Show that for every hyperplane $k$ disjoint from $h$, the hyperplanes crossing $k$ are the same as the hyperplanes crossing $h$. 

$h$ - an unflippable halfspace
Step 1 together with the Endometry Lemma gives:

A hyperplane crosses $h$ if it crosses $g h$. For every hyperplane translate $g h c h$.  

Step 2. For every hyperplane translate $g h$.  

Step 1 together with the Endometry Lemma gives:

Exercise (Endometry Lemma): If $M$ is a proper coocompact metric space and $f: M \rightarrow M$ is an isometry, then $f$ is onto.
So we have accomplished our goal: the hyperplanes that cross \( h \) are the same as hyperplanes that cross a hyperplane disjoint from \( h \).
Conclusion: We can now write $H$ as the disjoint union

$\mathcal{H} \cap \mathcal{H} = \mathcal{H}$

and every hyperplane in $\mathcal{H}$ intersects every hyperplane in $\mathcal{H}$. Let

$\mathcal{H} \cap \mathcal{H} = \mathcal{H}$

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$\mathcal{H} \cap \mathcal{H} = \mathcal{H}$
So $T = \mathbb{X}$ is a line.

Halfspace: $\text{Stab}(T) \leq G$ acts on $T$ properly and cocompactly and has an unflippable halfspace.

There is a copy $T$ of $\mathbb{X}$ which appears in $\mathbb{X}$ as a maximal intersection of hyperplanes meeting $h$.

$\mathbb{X}$ is a tree.
Double Skewering

 Lemma.

 If both $h$ and $k$ are flipppable flip twice

 1. If both $h$ and $k$ are flipppable flip twice

 Proof.

 If either $h$ or $k$ are unflipppable then by the Flipping Lemma we have that $g$ such that $gk \subset h$. Every for every pair of nested halfspaces $h \subset k$, there is some element of $g \in G \odot G \times G$ cocompactly and essentially then Double Skewering.
Lemma. If $h$ and $k$ are intersecting hyperplanes, then all four complementary regions (sectors) of $h \cup k$ contain a hyperplane.
Proof.

Step 1. Assume there exists some hyperplane disjoint from \( h \) and \( k \).
Step 2. Assume that every hyperplane either crosses $h$ or crosses $k$.

\[ (\mathcal{X} - \mathcal{H}) - \mathcal{H} = \mathcal{Y} \]

\[ \{ \mathcal{X} \in \pi \} = \mathcal{Y} \]

\[ \{ \mathcal{H} \in \pi \} = \mathcal{H} \]

\[ \{ \mathcal{Y} \in \pi \} = \mathcal{Y} \]

\[ \{ \mathcal{X} \in \pi \} = \mathcal{X} \]

Show that \( \mathcal{X} \cup \mathcal{H} \cup \mathcal{H} \) is a transverse decomposition of \( \mathcal{H} \).
Corollary. $X$ irreducible and not a line $\Rightarrow G$ has a free subgroup of rank 2.

Application: Tits Alternative
Proof of Rank Rigidity

1. Consider a maximal collection of intersecting hyperplanes having diagonally opposite sectors containing hyperplanes.

2. Find $g$ skewering those hyps.

3. Axis for $g$ bounds a half-flat.

4. Find a hyp crossing all of these.

5. Use Sector Lemma to contradict maximality.

I. Consider a maximal collection of intersecting hyperplanes having diagonally opposite sectors containing hyperplanes.