Lecture 2: Cubulations

Pocsets → CAT(0) Cube Complexes

Σ - locally finite, finite width pocset

An ultrafilter on Σ is a subset $\alpha \subset \Sigma$ satisfying

- Choice: For every $A \in \Sigma$, $A \in \alpha$ or $A^* \in \alpha$ (not both)
- Consistency: $A \subset \alpha$ and $A \subset B \Rightarrow B \in \alpha$

An ultrafilter α is said to satisfy the descending chain condition (DCC) if every descending chain
$A_1 > A_2 > \ldots$ in α terminates.

Examples

Tree	squaring of R^2	family of lines in R^2

- $X^{(0)}$ - DCC vertices
- $X^{(1)}$ - join α and β if they differ on a single choice:

$$\beta = (\alpha; A) \equiv (\alpha - \{A\}) \cup \{A^*\}$$

Exercise. $(\alpha; A)$ is a vertex $\Leftrightarrow$ A is minimal in α

- $X^{(2)}$ - attach squares to all 4-cycles in $X^{(1)}$

Exercise. α, $(\alpha; A)$, $(\alpha; B)$, $(\alpha; A,B)$ are vertices

$$\Updownarrow$$

$A, B \in \alpha$ are minimal and transverse

- $X^{(n)}$ - attach n-cubes wherever you can.

Exercise. The vertex α and $A_1,\dots,A_n \in \alpha$ span a cube

$$\Updownarrow$$

$$A_1,\dots,A_n \in \alpha \text{ are minimal and transverse}$$

Exercise. X is CAT(0)

Remark. Dimension of X = width of Σ

Remark. If pocset came with a group action, then the cube complex inherits an action.

(Wise) Small Cancellation Groups

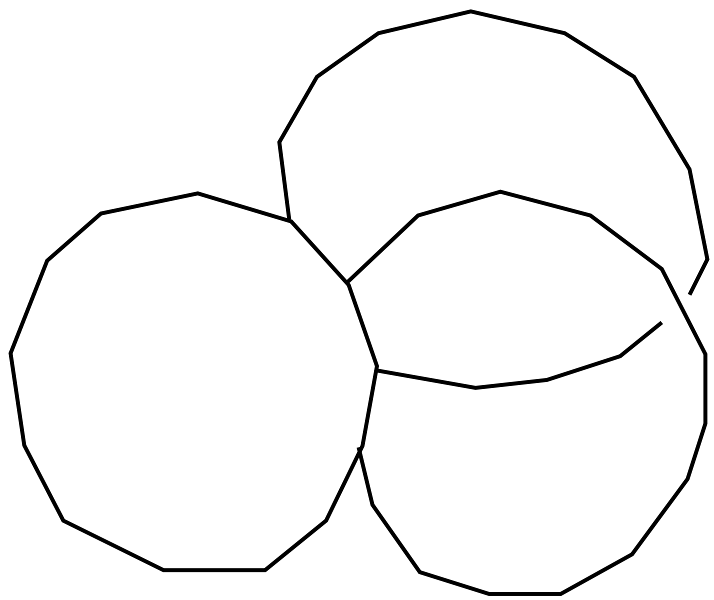
$G = \langle S \mid R \rangle$ where S, R are finite
(R cyclically closed)

A <u>piece</u> is a word that appears twice in R.

G is $C'(1/n)$ if the length of any piece is less than $1/n$ the length of any relator in which it appears.

<u>Gromov:</u> $C'(1/6)$ groups are hyperbolic.

Build walls in the universal cover of the presentation complex.



Draw Wise-tracks across cells.

Coxeter Groups (Niblo-Reeves)

For $1 \leq i \neq j \leq n$ we start with some numbers

$$m_{ij},\quad 2 \leq m_{ij} \leq \infty,\ m_{ij}=m_{ji}$$

Define $G=\langle s_1,\ldots,s_n \mid s_i^2, (s_is_j)^{m_{ij}} \rangle$

Build the presentation 2-complex and then collapse
each s_i^2 - bigon to an edge and each complex for a pair (i,j) to an m_{ij} - gon.

Draw tracks across cells.

Codimension 1 subgroups

$H < G$ finitely generated

If $Cay(G)/H$ has more than one end then we say that
H has <u>codimension 1</u> in G.

$p: Cay(G) \to Cay(G)/H$

$$p^{-1}(A) \to A$$

$\Omega = \{gA\} \cup \{gA^*\}$

$(Cay(G), \Omega)$ is a discrete space with walls

Cocompactness and Properness Criteria

(Gitik–Mitra–Rips–S): G hyperbolic group, H quasiconvex codimension 1 subgroup. Then the action on the associated cube complex is cocompact.

Warning (Rubinstein–Wang): There exists a closed 3-manifold with a surface subgroup such that the lifts the surface to the universal cover is a pattern of planes any two of which intersect.

$\Downarrow$

Cube complex is infinite dimensional!

<u>(Hruska-Wise):</u> $G \circlearrowright K$ by isometries, properly.
(K,Ω) a discrete wall space.
$d(x,gx) \to \infty \Rightarrow |\{\text{walls separating } x \ \& \ gx\}| \to \infty$

Then the action on the associated cube complex is proper.

<u>(Bergeron-Wise):</u> G hyperbolic such that quasi-convex subgroups separate points on the boundary. Then G acts properly and cocompactly on a CAT(0) cube complex

Application: hyperbolic 3-manifolds

(Kahn-Markovic): $G \circlearrowright H^3$ properly cocompactly
For every $\varepsilon > 0$ and every circle C in $S^2 = \partial H^3$,
there exists a surface subgroup in H whose limit set lies in the ε-neighborhood of C.

$\Downarrow$

The fundamental group of every hyperbolic 3-manifold acts properly and cocompactly on a CAT(0) cube complex

Duality of constructions

Finite dimensional CAT(0) cc's		Finite width Pocsets
X	$\longrightarrow$	$\mathcal{H}(X)$ - halfspaces
$X(\Sigma)$	$\longleftarrow$	Σ

(Roller): These constructions are dual to one another.

$$X(\mathcal{H}(X))=X \quad \text{and} \quad \mathcal{H}(X(\Sigma))=\Sigma$$

Exercise: Show this. (Also think about infinite dimensional complexes and infinite width pocsets.)

Application: Recognizing Products

A decomposition of X as a product $X = X_1 \times X_2$

$$\updownarrow$$

A decomposition $\hat{H} = \hat{H}_1 \cup \hat{H}_2$ where every hyperplane in $\hat{H}_1$ crosses every hyperplane in $\hat{H}_2$

Proof:

Corollary: Every finite dimensional CAT(0) cube complex admits a canonical decomposition as a product of irreducibles.