And what's it all good for

Groups which act on them

CAT(0) Cube Complexes
A CAT(0) cube complex is a 1-connected NPC cube complex. 

The link of each vertex is a flag complex (no missing simplices) •

The 1-skeleton of the link of a vertex is a simplicial graph (no 1,2-gons) •

unit Euclidean cubes by isometries of their faces so that:

A non-positively curved (NPC) cube complex is a cell complex built out of

NPC and CAT(0) cube complexes

Lecture 1: Introduction
Bridson: the natural piecewise path metric on a CAT(0) cube complex is indeed a metric.

Groves: With this metric, a CAT(0) cube complex is locally CAT(0).

Leary (appendix to "A metric Kan-Thurston theorem"): generalized to the infinite dimensional case.

Remark: The universal cover of a NPC complex is CAT(0).

Groups

Fundamental groups of NPC groups (and more generally, groups which act properly and cocompactly on CAT(0) cube complexes) are called cubed.
<table>
<thead>
<tr>
<th>A product</th>
<th>A graph</th>
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<tr>
<td>A Salvetti complex</td>
<td>A surface (genus &gt; 0)</td>
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**Exercise.** Show that a CAT(0) square complex is a product of trees if and only if the link of each vertex is a complete bipartite graph.

**Example.** (Janzen-Wise) Four square example of free cocompact quotient of a tree x tree which is not finitely covered by graph x graph. (Burger-Mozes have examples which are simple groups.)

![Diagram of square complexes](image)

**Exercise.** Say something about $G = \langle a, b, c \mid a^2 b^2 c^2 bac \rangle$
Definition: A map \( f: X \to Y \) between cube complexes is said to be a cubical map if it sends cubes to cubes and on each cube factors through a natural projection.

Exercise: Let \( f: X \to Y \) be a cubical map such that

- \( f \) is locally injective
- \( f(lk(v)) \) is a full subcomplex of \( lk(f(v)) \)

Show that

- \( X \) is NPC
- \( f_*: \pi_1(X) \to \pi_1(Y) \) is injective

*Hint: Cartan-Hadamard*
Hyperplanes

X - NPC cube complex

E - edges of X

Equivalence relation on E: ~ generated by

e ∼ f

Hyperplane dual to equivalence class:

Hyperplanes
Exercise. Find an infinite NPC square complex with two embedded hyperplanes.

Exercise. Find a NPC square complex with a single hyperplane.

Exercise. Find a NPC square complex with a single hyperplane.

Previously:

Exercise. Understand what hyperplanes look like in the examples discussed.
Fundamental hyperplane facts

• Embeddedness: A hyperplane in a $\text{CAT}(0)$ cube complex is embedded (the carrier of the hyperplane is a convex subcomplex)

• Separation: Each hyperplane separates the cube complex into two components

• Helly: Every collection of pairwise intersecting hyperplanes intersects

• Hereditary CAT(0)ness: Each hyperplane in a $\text{CAT}(0)$ cube complex is a $\text{CAT}(0)$ cube complex
Application: No f.g. torsion groups

Theorem. \( G \) finitely generated and \( G \circlearrowright X \) a finite-dimensional without a global fixed point. Then \( G \) is not a torsion group.

Lemma (Exercise). A geodesic 1-skeleton path between vertices crosses every hyperplane separating the vertices exactly once.

Proofs:
Observations:
1. \( \mathcal{H} \) is a poset under inclusion
2. The poset \( \mathcal{H} \) is locally finite
3. \( \mathcal{H} \) has an order reversing involution

Notation:
\( \mathcal{H} \) - half-spaces
\( \overset{\wedge}{h} \) - half-space
\( \overset{\wedge}{h} \) - hyperplane
\( \overset{\ast}{h} \) - opposite half-space
A pocset \( \Sigma \) is a poset with an order-reversing involution \( \ast: \Sigma \to \Sigma \).

Two elements \( A, B \subseteq \Sigma \) are said to be transverse if
\( A \neq B, A \neq B^\ast, A^\ast \neq B \) and \( A^\ast \neq B^\ast \).

A pocset \( \Sigma \) is said to be of width \( n \) if the size of the largest collection of mutually transverse elements is \( n \).

We will call the pocset \( \Sigma \) locally finite if it is the poset \( \mathbb{Z} \) is locally finite.
Spaces with walls (Nica, Chatterji-Niblo)

Example of a pocset: A space with walls is simply

Exercise: (prove, disprove or salvage if possible)

A space with walls is called discrete if for any two elements a, b ∈ S, there are finitely many $A \in \Omega$ satisfying $a \in A$ and $b \notin A$.

A space with walls is discrete if and only if the associated pocset is locally finite.

$\Omega$ - a collection of subsets of S closed under complementation

$S$ - a set

Example of a pocset: A space with walls is simply
| Surface in a 3-manifold | Curve on a surface | Lines in $\mathbb{R}^2$ |

Examples: