CAT(0) Cube Complexes

Groups which act on them

And what's it all good for

Lecture 1: Introduction

NPC and CAT(0) cube complexes

A non-positively curved (NPC) cube complex is a cell complex built out of unit Euclidean cubes by isometries of their faces so that:

- The 1-skeleton of the link of a vertex is a simplicial graph (no 1,2-gons)

- The link of each vertex is a flag complex (no missing simplices)

A CAT(0) cube complex is a 1-connected NPC cube complex.

Remarks

- Bridson: the natural piecewise path metric on a CAT(0) cube complex is indeed a metric.

- Gromov: With this metric, a CAT(0) cube complex is CAT(0). An NPC complex is locally CAT(0).

- Leary (appendix to "A metric Kan-Thurston theorem"): generalized to the infinite dimensional case.

- The universal cover of a NPC complex is CAT(0).

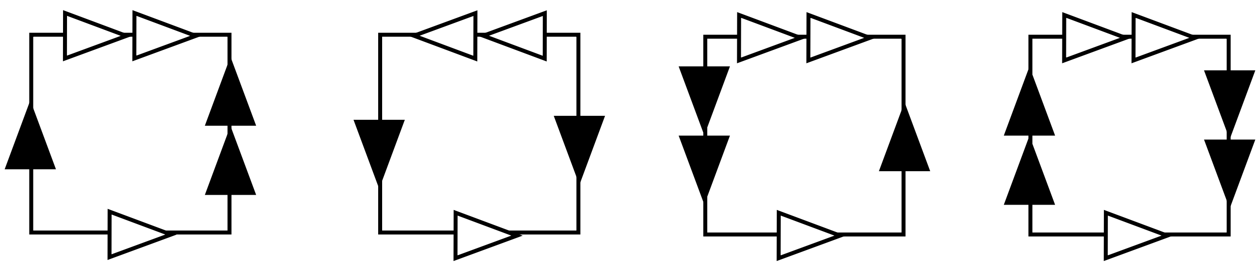
- Fundamental groups of NPC groups (and more generally, groups which act properly and cocompactly on CAT(0) cube complexes) are called cubed groups.

Examples

A graph	A surface (genus > 0)
A product	A Salvetti complex

Exercise. Show that a CAT(0) square complex is a product of trees if and only if the link of each vertex is a complete bipartite graph.

Example. (Janzen-Wise) Four square example of
free cocompact quotient of a tree x tree which is not finitely covered by graph x graph. (Burger-Mozes have examples which are simple groups.)



Exercise. Say something about $G = \langle a,b,c \mid a^2b^2c^2bac \rangle$

Maps

Definition: A map $f: X \to Y$ between cube complexes is said to be a cubical map if it sends cubes to cubes and on each cube factors through a natural projection.

Exercise: Let $f: X \to Y$ be a cubical map such that

- f is locally injective
- $f(\mathrm{lk}(v))$ is a full subcomplex of $\mathrm{lk}(f(v))$.

Show that

- X is NPC
- Show that $f_*: \pi_1(X) \to \pi_1(Y)$ is injective

Hint: Cartan-Hadamard

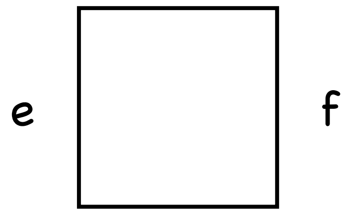
Hyperplanes

X - NPC cube complex

E - edges of X

Equivalence relation on E: ~ generated by

$e \sim f \quad \Leftrightarrow$



Hyperplane dual to
equivalence class:

Hyperplanes

Exercise. Understand what hyperplanes look like in the examples discussed previously.

Exercise. Find a NPC square complex with a single hyperplane.

Exercise. Find an infinite NPC square complex with two embedded hyperplanes.

Fundamental hyperplane facts

- Embeddedness: A hyperplane in a CAT(0) cube complex is embedded (the carrier of the hyperplane is a convex subcomplex)

- Separation: A hyperplane in CAT(0) cube complexes separates it into two components

- Hereditary CATness: Each hyperplane is a CAT(0) cube complex

- Helly: Every collection of pairwise intersecting hyperplanes intersects

Application: No f.g. torsion groups

Theorem. G finitely generated and $G \circlearrowright X$ a finite-dimensional without a global fixed point. Then G is not a torsion group.

Lemma (Exercise). A geodesic 1-skeleton path between vertices crosses every hyperplane separating the vertices exactly once.

Proofs:

Notation

$\mathcal{H}$ - half-spaces

$\hat{\mathcal{H}}$ - hyperplanes

h - half-space

$\hat{h}$ - hyperplane

h^* - opposite half-space

Observations:

1. $\mathcal{H}$ is a poset under inclusion
2. The poset $\mathcal{H}$ is locally finite
3. $\mathcal{H}$ has an order reversing involution

Pocsets

A pocset $\Sigma = (\Sigma, <, *)$ is a poset with an order-reversing involution $*: \Sigma \to \Sigma$.

Two elements $A, B \subset \Sigma$ are said to be transverse

if $A \nless B$, $A^* \nless B$, $A \nless B^*$ and $A^* \nless B^*$

Σ is said to be of width n if the size of the largest collection of mutually transverse elements is n.

We will call the pocset locally finite if it is the poset Σ is locally finite.

<u>Spaces with walls (Nica, Chatterji-Niblo)</u>

Example of a pocset: A <u>space with walls</u> is simply

S - a set

Ω - a collection of subsets of S closed under complementation

$*$ - $A \to S-A$

A space with walls is called <u>discrete</u> if for any two elements a,b \in S, there are finitely many $A \in \Omega$ satisfying $a \in A$ and $b \notin A$.

<u>Exercise.</u> (prove disprove or salvage if possible)
A space with walls is discrete if an only if the associated pocset is locally finite.

Examples:

Lines in R^2	Curve on a surface	Surface in a 3-manifold