LECTURES ON LATTICES

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1. Lecture 1, a brief overview on the theory of lattices

Let G be a locally compact group equipped with a left Haar measure μ, i.e. a Borel regular measure which is finite on compact, positive on open and invariant under left multiplications — by Haar's theorem such μ exists and is unique up to normalization. The group G is called unimodular if μ is also right invariant, or equivalently if it is symmetric in the sense that $\mu(A) = \mu(A^{-1})$ for every measurable set A. Note that G is compact iff $\mu(G) < \infty$.

For example:

- Compact groups, Nilpotent groups and Perfect groups are unimodular.
- The group of affine transformations of the real line is not unimodular.

A closed subgroup $H \leq G$ is said to be *co-finite* if the quotient space G/H admits a non-trivial finite G invariant measure. A *lattice* in G is a co-finite discrete subgroup. A discrete subgroup $\Gamma \leq G$ is a lattice iff it admits a finite measure fundamental domain, i.e. a measurable set Ω of finite measure which form a set of right cosets representatives for Γ in G. We shall denote $\Gamma \leq_L G$ to express that Γ is a lattice in G.

Exercise 1. *If G admits a lattice then it is unimodular.*

We shall say that a closed subgroup $H \leq G$ is *uniform* if it is cocompact, i.e. if G/H is compact.

Exercise 2. *A uniform discrete subgroup $\Gamma \leq G$ is a lattice. Note that if $G = SL_2(\mathbb{R})$ and H is the Boral subgroup of upper triangular matrices, then H is uniform but not co-finite.*

Examples 3.
1. *If G is compact every closed subgroup is cofinite. The lattices are the finite subgroups.*
2. *If G is abelian, a closed subgroup $H \leq G$ is cofinite iff it is cocompact.*
3. *Let G be the Heisenberg group of 3×3 upper triangular unipotent matrices over $\mathbb{R}$ and let $\Gamma = G(\mathbb{Z})$ be the integral points. Then Γ is a cocompact lattice in G.*
4. *Let T be a k regular tree equipped with a k coloring of the edges s.t. neighboring edges have different colors. Let $G = Aut(T)$ be the group of all automorphisms of T and let Γ be the group of those automorphisms that preserve the coloring. Then Γ is a uniform lattice in G.*
5. *$SL_n(\mathbb{Z})$ is a non-uniform lattice in $SL_n(\mathbb{R})$.*

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(6) *Let Σ_g be a closed surface of genus $g \geq 2$. Equip Σ_g with a hyperbolic structure and fix a base point and a unit tangent vector. The action of the fundamental group $\pi_1(\Sigma_g)$ via Deck transformations on the universal cover $\mathbb{H}^2 = \tilde{\Sigma}_g$ yields an embedding of $\pi_1(\Sigma_g)$ in $PSL_2(\mathbb{R}) \cong Isom(\mathbb{H}^2)^\circ$ and the image is a uniform lattice.*

1.1. **Lattices simulates their ambient group in many ways.** Here are few example of this phenomena: Let G be a l.c. group and $\Gamma \leq_L G$ a lattice.

(1) G is amenable iff Γ is amenable.
(2) G has property (T) iff Γ has property (T).
(3) Margulis' normal subgroup theorem: If G is a center free higher rank simple Lie group (e.g. $SL_n(\mathbb{R})$ for $n \geq 2$) then Γ is just infinite, i.e. has no infinite proper quotients.
(4) Borel density theorem: If G is semisimple real algebraic then Γ is Zariski dense.

1.2. **Some basic properties of lattices.**

Lemma 1.1 (Compactness criterion)**.** *Suppose $\Gamma \leq_L G$, let $\pi : G \to G/\Gamma$ be the quotient map and let $g_n \in G$ be a sequence. Then $\pi(g_n) \to \infty$ iff there is a sequence $\gamma_n \in \Gamma \setminus \{1\}$ such that $g_n\gamma_n g_n^{-1} \to 1$. In this case we shall say that the $\{\gamma_n\}$ is asymptotically unipotent, and that Γ has an approximated unipotent.*

Proof. If $\pi(g_n)$ does not go to infinity then a subsequance $\pi(g_{n_k})$ converges to some $\pi(g_0)$. Let W be an identity neighborhood which intersects Γ^{g_0} trivially, and let V be a symmetric identity neighborhood satisfying $V^3 \subset W$. For sufficiently large k we have $\pi(g_0 g_k^{-1}) \in \pi(V)$ which implies that Γ^{g_k} intersects V trivially.

Conversely, suppose that $\pi(g_n) \to \infty$. Let W be a an identity neighborhood in G and let V be a relatively compact symmetric identity neighborhood satisfying $V^2 \subset W$. Let K be a compact subset of G such that $\text{vol}(\pi(K)) > \text{vol}(G/\Gamma) - \mu(V)$. Since $\pi(G_n) \to \infty$, there is n_0 such that $n \geq n_0$ implies that $\pi(Vg_n) \cap \pi(K) = \emptyset$. The volumes inequality above then implies that $\text{vol}(\pi(Vg_n)) < \text{vol}(V)$ and we conclude that Vg_n is not injected to the quotient, i.e. that $Vg_n \cap Vg_n\gamma \neq \emptyset$ for some $\gamma \in \Gamma \setminus \{1\}$.

□

Corollary 1.2. *Γ admits approximated unipotents iff it is non-uniform.*

Lemma 1.3 (Recurrence)**.** *Let $\Gamma \leq_L G$, let $g \in G$ and let $\Omega \subset G$ be an open set. Then $\Omega g^n \Omega^{-1} \cap \Gamma \neq \emptyset$ infinitely often.*

Proof. This is immediate from Poincare recurrence theorem. □

Exercise 4. *Let $\Gamma \leq_L SL_n(\mathbb{R})$. Deduce from the last lemma that Γ admits regular elements and that $Span(\Gamma) = M_n(\mathbb{R})$.*

Proposition 1.4. *Let $\Gamma \leq_{UL} G$ (a uniform lattice) and $\gamma \in \Gamma$. Let $C_G(\gamma)$ be the centralizer of γ in G. Then $\Gamma \cap \mathbb{C}_G(\gamma)$ is a uniform lattice in $C_G(\gamma)$.*

Proof. There are two ways to prove this, one by constructing a compact fundamental domain for $\Gamma \cap C_G(\gamma)$ in $C_G(\gamma)$ and one by showing that the projection of $C_G(\gamma)$ to G/Γ is closed. Let us describe the first approach.

Let Ω is a relatively compact fundamental domain for Γ in G. For $h \in C_G(\delta)$ we can express h as $\omega\delta$ with $\omega \in \Omega$ and $\delta \in \Gamma$, so $\delta = \omega^{-1}h$. Thus we see that $\delta\gamma\delta^{-1}$ belongs to the relatively compact set $\Omega^{-1}\gamma\Omega$. Let $\delta_1, \dots, \delta_m \in \Gamma$ be chosen such that $\delta_i\gamma\delta_i^{-1}$, $i = 1, \dots m$ exhaust the finite set $\Omega^{-1}\gamma\Omega \cap \Gamma$. It is an exersize to see that $\cup_{i=1}^m \Omega\delta_i \cap C_G(\gamma)$ is a fundamental domain for $\Gamma \cap C_G(\gamma)$ in $C_G(\gamma)$. $\square$

Exercise 5. *Show that if $\Gamma \leq_{UL} SL_n(\mathbb{R})$ then Γ admits a diagonalized subgroup isomorphic to $\mathbb{Z}^{n-1}$.*

1.3. **Arithmeticity.** One of the highlights of the theory of lattices is the connection with arithmetic groups. This is illustrated in the following theorems:

Theorem 1.5 (Borel–Harish-Chandra)**.** *Let $\mathbb{G}$ be an algebraic group defined over $\mathbb{Q}$ which has no $\mathbb{Q}$ characters. Then $\mathbb{G}(\mathbb{Z}) \leq_L \mathbb{G}(\mathbb{R})$. Furthermore, $\mathbb{G}(\mathbb{Z}) \leq_{UL} \mathbb{G}(\mathbb{R})$ iff $\mathbb{G}$ has no $\mathbb{Q}$ co-characters.*

Definition 1.6. Let G be a Lie group. We shall say that a subgroup $\Gamma \leq G$ is arithmetic if there is a $\mathbb{Q}$ algebraic group $\mathbb{H}$ and a surjective homomorphism with compact kernel $f : \mathbb{H}(\mathbb{R}) \to G$ such that $f(\mathbb{H}(\mathbb{Z}))$ contains Γ as a subgroup of finite index.

Example 1.7. Let $f(x, y, z) = x^2 + y^2 - \sqrt{2}z^2$, and consider the $\mathbb{Q}[\sqrt{2}]$-group $\mathbb{G} = \mathrm{SO}(f)$ and the subgroup $\Gamma = \mathbb{G}(\mathbb{Z}[\sqrt{2}])$. Let $\mathbb{H} = \mathrm{Res}|_{\mathbb{Q}}^{\mathbb{Q}[\sqrt{2}]}\mathbb{G}$ and let $H = \mathbb{H}(\mathbb{R})$. Then $H \cong \mathrm{SO}(2, 1) \times SO(3)$ and Γ is isomorphic to $\mathbb{H}(\mathbb{Z})$. Thus Γ is an arithmetic lattice in $\mathrm{SO}(2, 1) \cong \mathrm{SL}_2(\mathbb{R})$ and since it has no unipotent it must be uniform. We conclude that Γ contains a subgroup of finite index isomorphic to a surface group.

Theorem 1.8 (Margulis)**.** *If G is a higher rank simple Lie group then every lattice is arithmetic.*

2. Lecture 2, on Jordan–Zassenhaus–Kazhdan–Margulis theorem

Given two subsets of a group $A, B \subset G$ we denote by $\{[A, B]\} := \{[a, b] : a \in A, b \in B\}$ the set of commutators. We define inductively $A^{(n)} := \{[A, A^{(n-1)}]\}$ where $A^{(0)} := A$.

By Edo–Iwasawa theorem every Lie group is locally isomorphic to a linear Lie groups. By explicate computation using sub-multiplicativity of matrix norms, one proves:

Lemma 2.1. *Every Lie group G admits an open identity neighborhood U such that $U^{(n)} \to 1$ in the sense that it is eventually included in every identity neighborhood.*

Exercise 6. *Let Δ be a group generated by a set $S \subset \Delta$. If $S^{(n)} = \{1\}$ for some n then Δ is nilpotent of class $\leq n$.*

Corollary 2.2. *If $\Delta \leq G$ is a discrete subgroup then $\langle \Delta \cap U \rangle$ is nilpotent.*

Furthermore, taking U sufficiently close to 1 we can even guarantee that every discrete group with generators in U is contained in a *connected* nilpotent group:

Theorem 2.3 (Zassenhaus (38) – Kazhdan-Margulis (68)). *Let G be a Lie group. There is an open identity neighborhood $\Omega \subset G$ such that every discrete subgroup $\Delta \leq G$ which is generated by $\Delta \cap U$ is contained in a connected nilpotent Lie subgroup of $N \leq G$. Moreover $\Delta \leq_{UL} N$.*

The idea is that near the identity the logarithm is well defined and two elements commute iff their logarithms commute. For a complete proof see [19] or [21].

A set Ω as in the theorem above is called a Zassenhaus neighborhood.

Since connected compact nilpotent groups are abelian, we deduce the following classical result:

Theorem 2.4 (Jordan). *For a compact Lie group K there is a constant $m \in \mathbb{N}$ such that every finite subgroup $\Delta \leq K$ admits an abelian subgroup of index $\leq m$.*

Let us make a short sidewalk before continuing the discussion about discrete groups. Suppose that K is a metric group. An ϵ-quasi morphism $f : F \to K$ from an abstract group F is a map satisfying $d(f(ab), f(a)f(b)) \leq \epsilon,\ \forall a, b \in F$. We shall say that K is quasi finite if for every ϵ there is an ϵ-q.m. from some finite group to K with ϵ-dense image. Relying on Jordan's theorem, Turing showed:

Theorem 2.5 (Turing (38)). *A compact Lie group is quasi finite iff it is a torus.*

Turing's theorem can be used to prove:

Theorem 2.6 ([12]). *A metric space is a limit of finite transitive spaces (in the Gromov–Hausdorff topology) iff it is homogeneous, its connected components are invers limit of tori and it admits a transitive compact group of isometries whose identity connected component is abelian.*

Corollary 2.7 (Answering a question of I. Benjamini). *S^2 cannot be approximated by finite homogeneous spaces. In fact the only manifolds that can be approximated are tori.*

Coming back from this short tour, let us present another classical result:

Theorem 2.8 (The Margulis lemma). *Let G be a Lie group acting by isometries with compact stabilizers on a Riemannian manifold X. Given $x \in X$ there are $\epsilon = \epsilon(x) > 0$ and $m = m(x) \in \mathbb{N}$ such that if $\Delta \leq G$ is a discrete subgroup which is generated by the set*

$$\Sigma_{\Delta,x,\epsilon} := \{\delta \in \Delta : d(\delta \cdot x, x) \leq \epsilon\}$$

then Δ admits a subgroup of index $\leq m$ which is contained in a connected nilpotent Lie group. Furthermore, if G acts transitively on X then ϵ and m are independent of x.

The basic idea is that the set $\{g \in G : d(g \cdot x, x \leq 1\}$ is compact and can be covered by boundedly many (say m) translations of an open symmetric set V such that V^2 is a

Zassenhaus neighborhood. Taking $\epsilon = 1/m$ one can prove the theorem. Again, for details see [21].

In the special case of $X = \mathbb{R}^n$ since homotheties commute with isometries it follows that $\epsilon = \infty$. Moreover, it is easy to verify that the connected nilpotent subgroups of the group $G = \mathrm{Isom}(\mathbb{R}^n) \cong O_n(\mathbb{R}) \ltimes \mathbb{R}^n$ are abelian. Using the fact that for any non-elliptic isometry of $\mathbb{R}^n$ one can decompose $\mathbb{R}^n$ invariantly and orthogonally to $\mathbb{R}^k \oplus \mathbb{R}^{n-k}$ where the isometry acts on the first factor by translation and on the second factor is by rotation, one deduces:

Theorem 2.9 (Bieberbach (11)— Hilbert's 18'th problem)**.** *Let Γ be a group acting properly discontinuously by isometries on $\mathbb{R}^n$. Then Γ admits a finite index subgroup isomorphic to $\mathbb{Z}^k$ ($k \le n$) which acts by translations on some k dimensional invariant subspace, and $k = n$ iff Γ is uniform. In particular, every crystallographic manifold is finitely covered by a torus.*

3. Lecture 3, on the geometry of locally symmetric spaces and some finiteness theorems

3.1. **Hyperbolic spaces.** Consider the hyperbolic space $\mathbb{H}^n$ and its group of isometries $G = \mathrm{Isom}(\mathbb{H}^n)$. Recall that $G^\circ \cong \mathrm{PO}(n,1)$ is a rank one simple Lie group. For $g \in G$ denote by $d_g(x) := d(g \cdot x, x)$ the displacement function of g at $x \in \mathbb{H}^n$. Let $\tau(g) = \inf d_g$ and $\min(g) = \{x :\in \mathbb{H}^n : d_g(x) = \tau(g)\}$. Note that d_g is a convex function which is smooth outside $\min(g)$.

The isometries of $\mathbb{H}^n$ splits to 3 types:

- *elliptic* — those that admit fixed points in $\mathbb{H}^n$.
- *hyperbolic* — isometries for which d_g attains a positive minima. In that case $\min(g)$ is a g invariant geodesic, called the axis of g.
- *parabolic* — isometries for which $\inf d_g = 0$ but have no foxed points in $\mathbb{H}^n$.

The first two types are called *semisimple.*

Exercise 7. *If $g, h \in G$ commute then*

- *if g is hyperbolic, then h is semisimple,*
- *if g and h are parabolics, they have the same fixed point at $\partial\mathbb{H}^n$.*

Exercise 8. *A discrete subgroup $\Delta \le G$ admits a common fixed point in $\mathbb{H}^n$ iff it is finite.*

It follows that a discrete group $\Gamma \le G$ acts freely on $\mathbb{H}^n$ iff it is torsion free.

Let $\Gamma \le G$ be a t.f. discrete subgroup, we denote by $M = \Gamma\backslash\mathbb{H}^n$ the associated complete hyperbolic manifold. Note that Γ is a lattice iff M has finite volume. We denote by $\mathrm{InjRad}(x)$ the injectivity radius at x. Let $\epsilon = \frac{1}{2}\epsilon(\mathbb{H}^n)$ be one half of the Margulis' constant, and let

$$M_{<\epsilon} = \{x \in M : \mathrm{InjRad}(x) < \epsilon/2\}, \text{ and } M_{\ge\epsilon} = \{x \in M : \mathrm{InjRad}(x) \ge \epsilon/2\}$$

be the ϵ-thin part and the ϵ-thick part of M.

Theorem 3.1 (The thick–thin decomposition)**.** *Supposing vol$(M) < \infty$, each connected component $M^\circ_{<\epsilon}$ of the thin part $M_{<\epsilon}$ is either*

- *a tubular neighborhood of a short closed geodesic, in which case $M^\circ_{<\epsilon}$ is homeomorphic to a ball bundle over a circle, or*
- *a cusp, in which case $M^\circ_{<\epsilon}$ is homeomorphic to $N \times \mathbb{R}^{>0}$ where N is some $(n-1)$-crystallographic manifold.*

The number of connected components of $M_{<\epsilon}$ is at most $C \cdot$ vol(M) for some constant $C = C(\mathbb{H}^n)$, and in case $n \geq 3$ the thick part $M_{\geq\epsilon}$ is connected.

Proof. Let $\tilde{M}_{<\epsilon}$ be the pre-image in $\mathbb{H}^n$ of the thin part of M. Observe that

$$\tilde{M}_{<\epsilon} = \cup_{\gamma\in\Gamma\setminus\{1\}}\{d_\gamma < \epsilon\}$$

is the union of the ϵ-sub-level sets. Note that a sub-level set of a hyperbolic isometry is a neighborhood of a geodesic and a sub-level set of a parabolic isometry is a neighborhood of the fixed point at infinity.

Consider $\alpha, \beta \in \Gamma \setminus \{1\}$ such that $\{d_\alpha \leq \epsilon\} \cap \{d_\beta \leq \epsilon\} \neq \emptyset$. By the Margulis lemma, $\alpha^{m!}, \beta^{m!}$ commute. Hence the exercise above implies that α and β are either both hyperbolic with the same axis or both parabolic with the same fixed point at infinity. It follows that a connected component of $\tilde{M}_{<\epsilon}$ is of the form $\cup_{\gamma\in I}\{d_\gamma \leq \epsilon\}$ where I consists either of hyperbolic elements sharing the same axis or of parabolic elements fixing a common point at infinity. In the first case, the discreteness of the torsion free group Γ implies that the element in I with minimal displacement generates a cyclic group containing I. In the second case, the elements in I preserve the horospheres around the fixed point at infinity, and the assumption vol$(M) < \infty$ implies that the quotient of such an horosphere by the group $\langle I\rangle$ must be compact.

Finally, in order to bound the number of components of $M_{<\epsilon}$ note that near the boundary of every component one can inject an ϵ-ball such that these balls are disjoint. □

Remark 3.2. An analog result holds (with almost the same proof) in every rank one symmetric space.

The thick-thin decomposition is an important ingredient in the proof of the following:

Theorem 3.3 ([7, 10])**.** *There is a constant $c = c(G)$ such that every t.f. lattice $\Gamma \leq_L G$ admits a presentation $\Gamma = \langle\Sigma|R\rangle$ with $|\Sigma|, |R| \leq c \cdot$ vol(G/Γ). Furthermore, unless $G \cong PSL_2(\mathbb{C})$ there is such a presentation in which the length of every relation is at most* 3.

Suppose $n > 3$ then gluing back the thin components to the thick part one by one and using Van–Kampen theorem, one sees that $\pi_1(M) \cong \pi_1(M_{\geq\epsilon})$. Now $M_{\geq\epsilon}$, being an ϵ-thick manifold (forget for a moment the boundary), can be covered by $\frac{\mathrm{vol}(M)}{\mathrm{vol}(B_{\epsilon/2})}$ balls of radius ϵ. Taking the nerve corresponding to that cover one gets a simplicial complex homotopic to $M_{\geq\epsilon}$, and it is not hard to verify that the fundamental group of that simplicial complex has presentation as above. Indeed the vertex degrees of the associated simplicial complex are uniformly bounded, and we can take a spanning tree and put a generator for each edge outside this tree and a relation for every 2-simplex. For a real proof of this theorem, not avoiding the boundary issue, check [10].

Remark 3.4. A similar theorem holds for every simple Lie group G, with few exceptions, but the proof is more complicated (see [10]).

3.2. **General symmetric spaces.**

Definition 3.5. A symmetric space is a complete Riemannian manifold X such that for every $p \in X$ there is an isometry i_p which fixes p and reflects the geodesics through p.

A symmetric space admits a canonical de-Rham decomposition $X = \prod X_i$ to irreducible factors. We shall say that X is of non-compact type if neither of the X_i are compact nor $\cong \mathbb{R}$. In that case $\text{Isom}(X)^\circ$ is a center-free semisimple Lie group without compact factors. Conversely, if G is a connected center-free semisimple Lie group without compact factors then G admits a, unique up to conjugacy, maximal compact subgroup K and G/K admits a canonical metric w.r.t which it is a symmetric space of non-compact type with isometry group whose identity component if G. Symmetric spaces of non-compact type are non-positively curved, i.e. they are CAT(0), and strictly negatively curved iff $\text{rank}(G) = 1$.

Example 3.6. As a model for the symmetric space of $G = \text{PSL}_n(\mathbb{R})$, denoted $P^1(n, \mathbb{R})$, we can take the space of all unimodular positive definite $n \times n$ matrices on which g acts by similarity: $g \cdot p := gpg^t$. The tangent space at I is the space of symmetric $n \times n$ matrices. The inner product at $T_I(P^1(n, \mathbb{R}))$ is given by $\langle X, Y \rangle := \text{trace}(XY)$, the geodesics through I are of the form $\exp(tX)$ and the curvature tensor at $X, Y \in T_I(P^1(n, \mathbb{R}))$ is given by $K(X, Y) = -\|[X, Y]\|$.

If Γ has torsion, the quotient space $\Gamma\backslash X$ is an orbifold. Dealing with the geometry of orbifolds is much more delicate. Still, using a basic Morse theory and the Margulis' lemma, one can prove:

Theorem 3.7 ([11])**.** *Let G be a connected semisimple Lie group without compact factors. There is a constant $C = C(G)$ such that $d(\Gamma) \leq C \cdot vol(G/\Gamma)$ for every discrete group $\Gamma \leq G$, where $d(\Gamma)$ denotes the minimal cardinality of a generating set.*

Let us explain the idea of the proof in the hyperbolic case (see also [2] for a detailed proof of this theorem in the rank one case). For $G = \text{PSL}_2(\mathbb{R})$, $X = \mathbb{H}^2$ the theorem can be deduced from the Gauss–Bonnet theorem, so let us assume that $n \geq 3$.

Lemma 3.8. *Let X be an irreducible symmetric space of dimension > 2. Let $g \in G = Isom(X)^\circ$ be a non-trivial element. Then* $\dim(X) - \dim(\min(g)) \geq 2$.

It follows that $\tilde{Y} = X \setminus \cup\{\min(\gamma) : \gamma \in \Gamma \setminus \{1\}\}$ is connected Γ-invariant subset of X. Let $Y = \Gamma\backslash\tilde{Y}$ be the image of $\tilde{Y}$ in M.

Let $f : \mathbb{R}^{>0} \to \mathbb{R}^{\geq 0}$ be a smooth function which tends to ∞ at 0, is strictly decreasing on $(0, \epsilon]$ and is identically 0 on $[\epsilon, \infty)$. Let $\Gamma_\circ = \{\gamma \in \Gamma \setminus \{1\} : \tau(\gamma) \leq \epsilon\}$. Define $\tilde{\psi} : \tilde{Y} \to \mathbb{R}$

as follows,:

$$\tilde{\psi}(x) = \sum_{\gamma \in \Gamma_\circ} f(d_\gamma(x) - \tau(\gamma)).$$

Note that $\tilde{\psi}$ is well defined since for every $x \in \tilde{Y}$ only finitely many of the summoneds are nonzero as Γ is discrete. Clearly $\tilde{\psi}$ is Γ-invariant and hence induces a map $\psi : Y \to \mathbb{R}^{\geq 0}$.

Lemma 3.9 (Main lemma)**.** *The gradient of ψ vanishes precisely where ψ vanishes.*

Proof. At any point $x \in \tilde{Y}$ we will find a tangent vector $\hat{n}_x$ at which the directional derivative of $\tilde{\psi}$ is nonzero. We distinguish between 3 cases. Let

$$\Sigma_x = \{\gamma \in \Gamma_\circ : f(d_\gamma(x)) \neq 0\}, \text{ let } \Delta_x = \langle \Sigma_x \rangle$$

and let N_x be a normal subgroup of finite index in Δ_x which is contained in a connected nilpotent Lie subgroup of G. In view of Selberg's lemma we may also suppose that N_x is torsion free. Let Z_x denote the center of N_x.

Case 1: Suppose first that Δ_x is finite. Let $y \in X$ be a fixed point for Δ_x and let $\hat{n}_x$ be the unit tangent at x to the geodesic ray $c : [0, \infty) \to X$ emanating from y through x. Thus $\hat{n}_x = \dot{c}(d(x, y))$. Since $x \in Y$ it follows that $d_\gamma(x) > 0,\ \forall \gamma \in \Sigma_x$, and since d_γ is a convex function we deduce that

$$\frac{d}{dt}|_{t=d(x,y)} d_g c(c(t)) > 0,$$

for every $\gamma \in \Sigma_x$ and hence

$$\partial\tilde{\psi}(x) \cdot \hat{n}_x = \frac{d}{dt}|_{t=d(x,y)} \tilde{\psi}(c(t)) = \sum_{\gamma \in \Sigma_x} f'(d_\gamma(x)) \frac{d}{dt}_{t=d(x,y)} d_\gamma(c(t)) < 0$$

since $\gamma \in \Sigma_x$ implies $d_\gamma(x) < \epsilon$ and f has negative derivative on $(0, \epsilon)$.

In the next two cases N_x and Z_x are nontrivial hence infinite.

Case 2: Suppose now that Z_x admits an hyperbolic element γ_0 and let A be the axis of γ_0. It follows that all elements of N_x preserve A and hence attain their minimal displacement on A. Thus $A = \cap \min_{\gamma \in N_x}(\gamma)$ and since N_x is normal it follows that A is also Δ_x invariant, and hence all elements in Δ_x attain their minimal displacement on A. Let $y = \pi_A(x)$ be the nearest point to x in A, let $c : [0, \infty)$ be the ray from y through x and let $\hat{n}_x$ be the tangent to c at x. Since A is convex $d_\gamma(x) \geq d_\gamma(y),\ \forall \gamma \in \Sigma_x$. Moreover it also follow from convexity and the fact that $\mathbb{H}^n$ admits no parallel geodesics that for every $\gamma \in \Sigma_x$ we have $d_\gamma(x) > d_\gamma(y)$. Thus one can proceed arguing as in case 1.

Case 3: We are left with the case that Z_x admits a parabolic element γ_0. Since Z_x is characteristic in N_x and N_x is normal, also Z_x is normal in Δ_x. Moreover, since N_x is of finite index in Δ_x the element γ_0 has only finitely many conjugate in Δ_x and they are all in Z_x. Denote these elements by $\gamma_0, \gamma_1, \ldots, \gamma_k$. All of them are parabolics and since they commute with each other, for every t the corresponding sublevel sets intersect nontrivially:

$$B_t = \cap_{i=0}^{k} \{p \in X : d_{\gamma_i}(p) \leq t\} \neq \emptyset.$$

Taking $t < \min\{d_{\gamma_i}(x) : i = 0, \dots, k\}$ we get a nonempty Δ_x-invariant closed convex set B_t not containing x. Taking $y = \pi_{B_t}(x)$ and proceeding as in the previous cases allows us to complete the proof. ☐

By the finiteness of the volume of M we deduce:

Lemma 3.10. *The map ψ is proper, i.e. $\psi^{-1}([0,a])$ is compact for every $a \in \mathbb{R}^{\geq 0}$.*

Proof. Suppose $a > 0$. If $\psi(x) \leq a$ and $\tilde{x}$ is a lift of x to $\mathbb{H}^n$ then for every $\gamma \in \Gamma \setminus \{1\}$ we have $f(d_\gamma(x)) \leq a$ implying that $d_\gamma(x) \geq f^{-1}(a)$. It follows that the injectivity radius of M at x is at least $f^{-1}(a)/2$. Thus $\psi^{-1}([0,a])$ is contained in the $f^{-1}(a)/2$-thick part of M. Since M has finite volume the last set is compact. ☐

Recall the following basic lemma from Morse theory:

Lemma 3.11 (Morse lemma)**.** *Let Q be a smooth manifold and and $\phi : Q \to \mathbb{R}^{\geq 0}$ a smooth proper map. If $\partial\phi \neq 0$ on $\phi^{-1}(a,b)$ for some $0 \leq a \leq b \leq \infty$ then $\phi^{-1}([0,a])$ is a deformation retract of $\phi^{-1}([0,b])$.*

Applying the lemma to $Q = Y$ and $\phi = \psi$ we deduce that $\psi^{-1}(0)$ is a deformation retract of Y. It follows that $\pi_1(Y) \cong \pi_1(\psi^{-1}(0))$. Note that since Γ acts freely on $\tilde{Y}$ and $Y = \Gamma\backslash\tilde{Y}$ it follows that Γ is a quotient of $\pi_1(Y)$. Hence the theorem will follow if we show:

Lemma 3.12. *$\pi_1(\psi^{-1}(0))$ is generated by $C \cdot vol(M)$ elements for some appropriate constant $C = C(\mathbb{H}^n)$.*

Proof. Let $\mathcal{F}$ be a maximal ϵ discrete subset of $\psi^{-1}(0)$. Since the $\epsilon/2$ balls centered at $\mathcal{F}$ are disjoint and injected

$$|\mathcal{F}| \leq \text{Const} \cdot \text{vol}(M).$$

Let U be the union of the ϵ balls centered at $\mathcal{F}$. Then $\psi^{-1}(0) \subset U \subset Y$ and since $\psi^{-1}(0)$ is a deformation retract of Y we see that $\pi_1(\psi^{-1}(0))$ is a quotient of $\pi_1(U)$. Finally U is homotopic to the simplicial complex corresponding to the nerve of the cover $\{B(f,\epsilon) : f \in \mathcal{F}\}$ and the complexity of the letter is clearly bounded by a constant times $|\mathcal{F}|$. ☐

This finishes the proof of the theorem in case $X = \mathbb{H}^n$. ☐

As an immediate consequence we deduce the following result which originally proved by Garland and Raghunathan in the rank one case and by Kazhdan in the higher rank case:

Corollary 3.13. *Every lattice in G is finitely generated.*

As another application we deduce:

Theorem 3.14 (Kazhdan–Margulis 68)**.** *Let G be as above. There is a positive constant v_0 such that for every lattice $\Gamma \leq_L G$, $vol(G/\Gamma) \geq v_0$.*

It can be shown that the minimal co-volume v_0 is attained, but in general it is very hard to get a good estimate of it.

4. Lecture 4, Rigidity and arithmeticity

Let Γ be a finitely generated group and G a topological group. By $\mathrm{Hom}(\Gamma, G)$ we denote the space of homomorphisms $\Gamma \to G$ with the point-wise topology. A map $f \in \mathrm{Hom}(\Gamma, G)$ is said to be locally rigid if the conjugacy class f^G contains a neighborhood of f. A subgroup $\Gamma \leq G$ is said to be locally rigid if the inclusion map $\Gamma \to G$ is locally rigid.

Theorem 4.1 (Local rigidity)**.** *(Margulis, Weil, Selberg, Calabi) Let G be a connected semisimple Lie group group not locally isomorphic to $PSL_2(\mathbb{R})$ or $PSL_2(\mathbb{C})$. Then every irreducible lattice is locally rigid. If G is locally isomorphic to $PSL_2(\mathbb{C})$ and $\Gamma \leq_L G$ then Γ is locally rigid iff it is uniform.*

A. Selberg who proved local rigidity for uniform lattices in $\mathrm{SL}_n(\mathbb{R}),\ n \geq 3$ observed that this implies:

Proposition 4.2. *Let G be a semisimple Lie group and Γ a locally rigid subgroup then the eigenvalues of $Ad(\gamma)$ are algebraic for every $\gamma \in \Gamma$.*

The local rigidity theorem is also an important ingredient in the proof of Wang's finiteness theorem:

Theorem 4.3 (Wang's finiteness theorem)**.** *Let G be a connected semisimple Lie group without compact factors not locally isomorphic to $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$. Then for every $v > 0$ there are only finitely many irreducible lattices $\Gamma \leq G$ with $vol(G/\Gamma) < v$.*

Next let us formulate Mostow's and Margulis' rigidity theorems:

Theorem 4.4 (Strong rigidity, Mostow)**.** *Let G be a semisimple Lie group without compact factors, and suppose that $\dim(G) > 3$. Let $\Gamma_1, \Gamma_2 \leq_L G$ be irreducible lattices then every isomorphism between Γ_1 and Γ_2 extends to an authomorphism of G.*

As a corollary, one deduce for instance that two finite volume complete n-hyperbolic manifolds are isometric iff their fundamental groups are isomorphic. Let us demonstrate another result whose proof relies on Mostow's theorem:

Theorem 4.5. *Let G be a semisimple Lie group without compact factors not locally isomorphic to $SL_2(\mathbb{R}), SL_2(\mathbb{C})$. Then the number of conjugacy classes of lattices of covolume $\leq v$ is at most v^{cv} where c is some constant depending on G.*

Theorem 4.6 (Margulis super-rigidity theorem)**.** *Let G be a semisimple Lie group without compact factors and suppose $rank_{\mathbb{R}}(G) \geq 2$. Let $\Gamma \leq_L G$ be an irreducible lattice. Let $\mathbb{H}$ be a center free simple algebraic group defined over a local field k and let $\rho : \Gamma \to \mathbb{H}(k)$ be a Zariski dense unbounded representation. Then ρ extends uniquely to a representation of G.*

We shall explain how super-rigidity implies arithmeticity:

Theorem 4.7. *Let G be a higher rank semisimple Lie group without compact factors. Then every irreducible lattice is arithmetic.*

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