LECTURE 3: PROPERTY (τ) AND EXPANDERS (PRELIMINARY VERSION)

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There are many excellent existing texts for the material in this lecture, starting with Lubotzky's monograph [11] and recent AMS survey paper [10]. For expander graphs and their use in theoretical computer science, check the survey by Hoory, Linial and Wigderson [6]. We give here a brief introduction.

I. Expander graphs

We start with a definition.

Definition 0.1. *(Expander graph) A finite connected k-regular graph $\mathcal{G}$ is said to be an ε-expander if for every subset A of vertices in $\mathcal{G}$, with $|A| \leqslant \frac{1}{2}|\mathcal{G}|$, one has the following isoperimetric inequality:*

$$|\partial A| \geqslant \varepsilon |A|,$$

where ∂A denotes the set of edges of $\mathcal{G}$ which connect a point in A to a point in its complement A^c.

The optimal ε as above is sometimes called the *discrete Cheeger constant* of the graph:

$$h(\mathcal{G}) = \inf_{A \subset \mathcal{G}, |A| \leqslant \frac{1}{2}|\mathcal{G}|} \frac{|\partial A|}{|A|},$$

Just as in Lecture 1, when we discussed the various equivalent definitions of amenability, it is not a surprise that this definition turns out to have a spectral interpretation.

Given a k-regular graph $\mathcal{G}$, one can consider the Markov operator (also called averaging operator, or sometimes Hecke operator in reference to the Hecke graph of an integer lattice) on functions on vertices on $\mathcal{G}$ defined as follows:

$$Pf(x) = \frac{1}{k} \sum_{x \sim y} f(y),$$

where we wrote $x \sim y$ to say that y is a neighbor of x in the graph.

This operator is easily seen to be self-adjoint on $\ell^2(\mathcal{G})$, which is a finite dimensional Euclidean space. Moreover it is a contraction, namely $||Pf||_2 \leqslant ||f||_2$ and hence its spectrum is real and contained in $[-1, 1]$. We can write the eigenvalues of P in decreasing

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order as $\mu_0 = 1 \geqslant \mu_1 \geqslant \ldots \geqslant \mu_{|\mathcal{G}|}$. The top eigenvalue μ_0 must be 1, because the constant function $\mathbf{1}$ is clearly an eigenfunction of P, with eigenvalue 1. On the other hand, since $\mathcal{G}$ is connected $\mathbf{1}$ is the only eigenfunction (up to scalars) with eigenvalue 1. This is immediate by the maximum principle (if $Pf = f$ and f achieve its maximum at x, then f must take the same value $f(x)$ at each neighbor of x, and this value spreads to the entire graph). Hence the second eigenvalue μ_1 is strictly less than 1.

Instead of P, we may equally well consider $\Delta := Id - P$, which is then a non-negative self-adjoint operator. This operator is called the *combinatorial Laplacian* in analogy with the Laplace-Beltrami operator on Riemannian manifolds.

$$\Delta f(x) := f(x) - \frac{1}{k}\sum_{x\sim y} f(y).$$

Its eigenvalues are traditionally denoted by $\lambda_0 = 0 < \lambda_1 \leqslant \ldots \leqslant \lambda_{|\mathcal{G}|}$ and :

$$\lambda_i(\mathcal{G}) = 1 - \mu_i(\mathcal{G}).$$

As promised, here is the connection between the spectral gap an the edge expansion.

Proposition 0.2. *(Discrete Cheeger-Buser inequality) Given a connected k-regular graph, we have:*

$$\frac{1}{2}\lambda_1(\mathcal{G}) \leqslant \frac{1}{k}h(\mathcal{G}) \leqslant \sqrt{2\lambda_1(\mathcal{G})}$$

The proof of this proposition follows a similar line of argument as the proof we gave in Lecture 1 of the Kesten criterion relating the Folner condition and the spectral radius of the averaging operator. See Lubotzky's book [11] for detailed derivation.

We note in passing that, since P is self-adjoint, the following holds:

$$||P||_{\ell^2_0} = \max_{i\neq 0} |\mu_i|$$

where ℓ^2_0 is the space of functions on $\mathcal{G}$ with zero average, and

$$\mu_1 = \sup\{\frac{\langle Pf, f\rangle}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\}$$

and hence

$$\lambda_1 = \inf\{\frac{\langle \Delta f, f\rangle}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\} = \frac{1}{k}\inf\{\frac{||\nabla f||_2^2}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\}.$$

Expander graphs have many very interesting applications in theoretical computer science (e.g. in the construction of good error correcting codes, see [6]). There typically one wants to have a graph of (small) bounded degree (i.e. k is bounded) but whose number of vertices is very large. For this it is convenient to use the following definition:

Definition 0.3. *(family of expanders) Let $k \geqslant 3$. A family $(\mathcal{G}_n)_n$ of k-regular graphs is said to be a family of expanders if the number of vertices $|\mathcal{G}_n|$ tends to $+\infty$ and if there is $\varepsilon > 0$ independent of n such that for all n*

$$\lambda_1(\mathcal{G}_n) \geqslant \varepsilon.$$

Although almost every random k-regular graph is an expander (Pinsker 1972), the first explicit construction of an infinite family of expander graphs was given using Kazhdan's property (T) and is due to Margulis [13] (see below Proposition 0.5).

Clearly an ε-expander graph of size N has diameter at most $O(\frac{1}{\varepsilon}\log|\mathcal{G}|)$. But more is true. A very important feature of expander graphs is the fact that the simple random walk on such a graph equidistributes as fast as could be towards the uniform probability distribution. This is made precise by the following proposition:

Proposition 0.4. *(Random walk characterization of expanders) Suppose $\mathcal{G}$ is a k-regular graph such that $|\mu_i(\mathcal{G})| \leqslant 1-\varepsilon$ for all $i \neq 0$, then there is $C = C(\varepsilon, k) > 0$ such that if $n \geqslant C \log|\mathcal{G}|$ then*

$$\max_{x,y} |\langle P^n \delta_x, \delta_y\rangle - \frac{1}{|\mathcal{G}|}| \leqslant \frac{1}{|\mathcal{G}|^{10}}.$$

Conversely for every $C > 0$ there is $\varepsilon = \varepsilon(C,k) > 0$ such that if the k-regular graph $\mathcal{G}$ satisfies

$$\max_{x} |\langle P^{2n} \delta_x, \delta_x\rangle - \frac{1}{|\mathcal{G}|}| \leqslant \frac{1}{|\mathcal{G}|^{10}},$$

for some $n \leqslant C \log|\mathcal{G}|$, then $\mathcal{G}$ satisfies $|\mu_i(\mathcal{G})| \leqslant 1-\varepsilon$ for all $i \neq 0$ (and in particular is an expander).

Here $\langle P^n \delta_x, \delta_y\rangle$ can be interpreted in probabilistic terms as the transition probability from x to y at time n, namely the probability that a simple (=equiprobable nearest neighbor) random walk starting at x visits y at time n.

The condition here that $||P|| = \max_{i\neq 0}|\mu_i(\mathcal{G})| \leqslant 1-\varepsilon$ is only slightly stronger than being an expander. The only difference is that we require the smallest eigenvalue $\mu_{\mathcal{G}}$ to be bounded away from -1 as well. In practice this is often satisfied and one can always get this by changing $\mathcal{G}$ into the induced k^2-regular graph obtained by connecting together vertices at distance 2 in $\mathcal{G}$ (which has the effect of changing P into P^2, hence squaring the eigenvalues).

The exponent 10 in the remainder term is nothing special and can be replaced by any exponent > 1.

Proof. The function $f_x := \delta_x - \frac{1}{|\mathcal{G}|}\mathbf{1}$ has zero mean on $\mathcal{G}$, hence

$$|\langle P^n \delta_x, \delta_y\rangle - \frac{1}{|\mathcal{G}|}| = |\langle P^n f_x, \delta_y\rangle| \leqslant ||P||^n ||f_x|| ||\delta_y|| \leqslant \sqrt{2}(1-\varepsilon)^n.$$

Now this is at most $1/|\mathcal{G}|$ as some as $n \geqslant C_\varepsilon \log|\mathcal{G}|$ for some $C_\varepsilon > 0$.

Conversely observe that $trace(P^{2n}) = \sum_{x\in\mathcal{G}}\langle P^{2n}\delta_x, \delta_x\rangle$, and hence summing the estimates for $\langle P^{2n}\delta_x, \delta_x\rangle$, we obtain

$$|trace(P^{2n}) - 1| \leqslant \frac{1}{|\mathcal{G}|^9},$$

But on the other hand $trace(P^{2n}) = 1 + \mu_1^{2n} + \ldots + \mu_{|\mathcal{G}|}^{2n}$, hence

$$\max_{i\neq 0} |\mu_i|^{2n} \leqslant \mu_1^{2n} + \ldots + \mu_{|\mathcal{G}|}^{2n} \leqslant \frac{1}{|\mathcal{G}|^9},$$

thus recalling that $|\mathcal{G}|^{1/\log|\mathcal{G}|} = e$, we obtain the desired upper bound on $\max_{i\neq 0}|\mu_i|$. □

This fast equidistribution property is usually considered as a feature of expander graphs, a consequence of the spectral gap. We will see in the last lecture, when explaining the Bourgain-Gamburd method, that the proposition can also be used in the reverse direction and be used to establish the spectral gap.

For more about random walks on finite graphs and groups and the speed of equidistribution (the cut-off phenomenon, etc) see the survey by Saloff-Coste [14].

II. Property (τ)

Margulis [13] was the first to construct an explicit family of k-regular expander graphs. For this he used property (T) through the following observation:

Proposition 0.5. *((T) implies (τ)) Suppose Γ is a group with Kazhdan's property (T) and S is a symmetric set of generators of Γ of size $k = |S|$. Let $\Gamma_n \leqslant \Gamma$ be a family of finite index subgroup such that the index $[\Gamma : \Gamma_n]$ tends to $+\infty$ with n. Then the family of Schreier graphs $\mathcal{S}(\Gamma/\Gamma_n, S)$ forms a family of k-regular expanders.*

Recall that the Schreier graph of a coset space Γ/Γ_0 associated to a finite symmetric generating set S of Γ is the graph whose vertices are the left cosets of Γ_0 in Γ and one connects $g\Gamma_0$ to $h\Gamma_0$ if there is $s \in S$ such that $g\Gamma_0 = sh\Gamma_0$.

Proof. The group Γ acts on the finite dimensional Euclidean space $\ell_0^2(\Gamma/\Gamma_n)$ of ℓ^2 functions with zero average on the finite set Γ/Γ_n. Denote the resulting unitary representation of Γ by π_n. Property (T) for Γ gives us the existence of a Kazhdan constant $\varepsilon = \varepsilon(S) > 0$ such that $\max_{s\in S}||\pi(s)v - v|| \geqslant \varepsilon||v||$ for every unitary representation π of Γ without invariant vectors. In particular, this applies to the π_n since they have no non-zero Γ-invariant vector. This implies that the graphs $\mathcal{G}_n := \mathcal{S}(\Gamma/\Gamma_n, S)$ are ε-expanders, because if $A \subset \mathcal{G}_n$ has size at most half of the graph, then $v := 1_A - \frac{|A|}{|\mathcal{G}|}\mathbf{1}$ is a vector in $\ell_0^2(\Gamma/\Gamma_n)$ and $||\pi_n(s)v - v||^2 = ||\pi_n(s)1_A - 1_A||^2 = |sA\Delta A|$, while $||v||^2 = 2|A|(1 - \frac{|A|}{|\mathcal{G}_n|}) \geqslant |A|$. In particular $|\partial A| \geqslant \varepsilon^2|A|$. □

So we see that Cayley graphs (or more generally Schreier graphs) of finite quotients of finitely generated groups can be yield families of expanders. This is the case for the family of Cayley graphs of $\mathrm{SL}_3(\mathbb{Z}/m\mathbb{Z})$ associated to the reduction mod m of a fixed generating set S in $\mathrm{SL}_3(\mathbb{Z})$. To characterize this property, Lubotzky introduced the following terminology:

Definition 0.6. *(Property (τ)) A finitely generated group Γ with finite symmetric generating set S is said to have* property (τ) *with respect to a family of finite index normal subgroups $(\Gamma_n)_n$ if the family of Cayley graphs $\mathcal{G}(\Gamma/\Gamma_n, S_n)$, where $S_n = S\Gamma_n/\Gamma_n$ is the projection of S to Γ/Γ_n, is a family of expanders. If the family $(\Gamma_n)_n$ runs over all finite index normal subgroups of Γ, then we say that Γ has* property (τ).

Proposition 0.5 above shows that every group with property (T) has property (τ). The converse is not true and property (τ) is in general a weaker property which holds more often. For example Lubotzky and Zimmer showed that an irreducible lattice in a semisimple real Lie group has property (τ) as soon as one of the simple factors of the ambient semisimple Lie group is of real rank at least 2 (and hence has property (T) by Kazhdan's theorem).

Property (τ) is stable under quotients and under passing to and from a finite index subgroup. In particular groups with property (τ) have finite abelianization, just as Kazhdan's groups.

Arithmetic lattices in semisimple algebraic groups defined over $\mathbb{Q}$ admit property (τ) with respect to the family of all congruence subgroups. Namely:

Theorem 0.7. *(Selberg, Burger-Sarnak, Clozel) Let $\mathcal{G} \subset \mathrm{GL}_d$ is a semisimple algebraic $\mathbb{Q}$-group, $\Gamma = \mathcal{G}(\mathbb{Z}) = \mathcal{G}(\mathbb{Q}) \cap \mathrm{GL}_d(\mathbb{Z})$ and $\Gamma_m = \Gamma \cap \ker(\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/m\mathbb{Z}))$, then Γ has property (τ) with respect to the Γ_m's.*

This property is also called the *Selberg property* because in the case of $\mathcal{G} = \mathrm{SL}_2$ it follows (see below) from the celebrated theorem of Selberg, which asserts that the non-zero eigenvalues of the Laplace-Beltrami laplacian on the hyperbolic surfaces of finite co-volume $\mathbb{H}^2/\ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}))$ are bounded below by a positive constant independent of m (in fact $\frac{3}{16}$). The general case was established by Burger-Sarnak and Clozel.

This connects property (τ) for lattices with another interesting feature of some lattices, namely the *congruence subgroup property.* This property of an arithmetic lattice asks that every finite index subgroup contains a congruence subgroup (i.e. a subgroup of the form $\mathcal{G}(\mathbb{Z}) \cap \ker(\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/m\mathbb{Z}))$. It is easy to see that if $\mathcal{G}(\mathbb{Z})$ has both the Selberg property and the congruence subgroup property, then it has property (τ) (with respect to all of its finite index subgroups). See the exercise sheet.

An interesting open problem in this direction is to determine whether or not lattices in $SO(n, 1)$ can have property (τ) or not. Lubotzky and Sarnak conjecture that they do not, and this would also follow from Thurston's conjecture that such lattices have a subgroup of finite index with infinite abelianization.

The link between Selberg's $\frac{3}{16}$ theorem and property (τ) is provided by the following general fact, which relates the combinatorial spectral gap of a Cayley (or Schreier) graph of finite quotients of the fundamental group of a manifold with the spectral gap for the analytic Laplace-Beltrami operator on the Riemannian manifold.

Recall that given a connected Riemannian manifold M the Laplace-Beltrami operator is a non-negative self-adjoint operator for L^2 functions with respect to the Riemannian volume measure and that if M is compact, its spectrum is discrete $\lambda_0(M) = 1 < \lambda_1(M) \leqslant \ldots$ (e.g. see [1]).

The fundamental group $\Gamma = \pi_1(M)$ acts freely and co-compactly on the universal cover $\widetilde{M}$ by isometries (for the lifted Riemannian metric on $\widetilde{M}$). Given a base point $x_0 \in \widetilde{M}$, the set

$$\mathcal{F}_M = \{x \in \widetilde{M}; d(x, x_0) < d(x, \gamma \cdot x_0) \ \forall \gamma \in \Gamma \setminus \{1\}\}$$

is a (Dirichlet) fundamental domain for the action of Γ on $\widetilde{M}$. Moreover the group Γ is generated by the finite symmetric set $S := \{\gamma \in \Gamma; \gamma\overline{\mathcal{F}_M} \cap \overline{\mathcal{F}_M} \neq \varnothing\}$. We can now state:

Theorem 0.8. *(Brooks [3], Burger [4]) Let M be a compact Riemannian manifold with fundamental group $\Gamma = \pi_1(M)$. Let S be the finite symmetric generating set of Γ obtained from a Dirichlet fundamental domain $\mathcal{F}_M$ as above. Then there are constants $c_1, c_2 > 0$ depending on M only such that for every finite cover M_0 of M*

$$c_1\lambda_1(M_0) \leqslant \lambda_1(\mathcal{G}(\Gamma/\Gamma_0, S)) \leqslant c_2\lambda_1(M_0),$$

where Γ_0 is the fundamental group of M_0 and $\mathcal{G}(\Gamma/\Gamma_0, S))$ the Schreier graph of the finite coset space Γ/Γ_0 associated to the generating set S.

We deduce immediately:

Corollary 0.9. *Suppose $(M_n)_n$ is a sequence of finite covers of M. Then there is a uniform lower bound on $\lambda_1(M_n)$ if and only if $\Gamma := \pi_1(M)$ has property (τ) with respect to the sequence of finite index subgroups $\Gamma_n := \pi_1(M_n)$.*

The proof consists in observing that the Schreier graph can be drawn on the manifold M_0 as a dual graph to the decomposition of M_0 into translates of the fundamental domain $\mathcal{F}_M$. The inequality on the left hand side is easier as one can use the interpretation in terms of Cheeger constants and given a set A of vertices with $|\partial A| \leqslant \varepsilon|A|$ one can look at the corresponding union of fundamental domains in M_0 and see that its boundary has small surface area compared to its volume. The other direction is a bit more involved and requires comparing the Rayleigh quotients $\frac{||\nabla f||}{||f||}$ of a function on M_0 with the combinatorial Rayleigh quotients of the function on the vertices of the graph obtained by averaging f over each fundamental domain. The result also extends to non-compact hyperbolic manifolds of finite co-covolume (see [2, Section 2] and [5, Appendix]).

For more on property (τ) we refer the reader to the book by Lubotzky and Zuk [12].

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