LECTURE 2: THE TITS ALTERNATIVE AND KAZHDAN'S PROPERTY (T) (PRELIMINARY VERSION)

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I. The Tits alternative.

A very interesting large class of groups is provided by the *linear groups*, namely the subgroups of $\mathrm{GL}_d(K)$, for some (commutative) field K. There are few general tools to study arbitrary finitely generated groups (often one has to resort to combinatorics and analysis as we did in Lecture 1 above for example). However for linear groups the situation is very different and a wide range of techniques (including algebraic number theory and algebraic geometry) become available.

Jacques Tits determined in 1972 which linear groups are amenable by showing his famous alternative:

Theorem 0.1. *(Tits alternative [18]) Let Γ be a finitely generated linear group (overs some field K). Then*

- *either Γ is virtually solvable (i.e. has a solvable finite index subgroup),*
- *or Γ contains a non-abelian free subgroup F_2.*

Remark. Virtually solvable subgroups of $\mathrm{GL}_d(K)$ have a subgroup of finite index which can be triangularized over the algebraic closure (Lie-Kolchin theorem).

In particular (since free subgroups are non-amenable and subgroups of amenable groups are amenable),

Corollary 0.2. *A finitely generated linear group is amenable if and only if it is virtually solvable.*

The proof of the Tits alternative uses a technique called "ping-pong" used to find generators of a non-abelian free subgroup in a given group. The basic idea is to exhibit a certain geometric action of the group Γ on a space X and two elements $a, b \in \Gamma$, the "ping-pong players" whose action on X has the following particular behavior:

Lemma 0.3. *(Ping-pong lemma) Suppose Γ acts on a set X and there are two elements $a, b \in \Gamma$ and 4 disjoint (non-empty) subsets A^+, A^-, B^+, and B^- of X such that*

- *a maps B^+, B^- and A^+ into A^+,*
- *a^{-1} maps B^+, B^- and A^- into A^-,*
- *b maps A^+, A^- and B^+ into B^+, and*
- *b^{-1} maps A^+, A^- and B^- into B^-.*

Then a and b are free generators of a free subgroup $\langle a, b\rangle \simeq F_2$ in Γ.

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Proof. The subset A^+ is called the attracting set for a and A^- the repelling set, and similarly for the other letters. Pick a reduced word w in a and b and their inverses. Say it starts with a. Pick a point p not in A^+ and not in the repelling set of the last letter of w (note that there is still room to choose such a p) Then the above ping-pong rules show that $w \cdot p$ belongs to A^+ hence is not equal to p. In particular w acts non trivially on X and hence is non trivial in Γ. $\square$

Remark. There are also other variant of the ping-pong lemma (e.g. it is enough that there are disjoint non-empty subsets A and B such that any (positive or negative) power of a sends B inside A and any power of b sends A inside B (e.g. take $A := A^+ \cup A^-$ and $B := B^+ \cup B^-$ above). But the above is the most commonly used in practice.

On Tits' proof. Tits' proof uses algebraic number theory and representation theory of linear algebraic groups to construct a local field ($\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$) K and an irreducible linear representation of Γ in $GL_m(K)$ whose image is unbounded. If Γ is not virtually solvable, one can take $m \geqslant 2$. Then he shows that one can change the representation (passing to an exterior power) and exhibit an element γ of Γ which is semisimple and has the property that both γ and γ^{-1} have a unique eigenvalue (counting multiplicity) of maximal modulus (such elements are called proximal elements). Then one considers the action of Γ on the projective space of the representation $X := \mathbb{P}(K^m)$ and observes that the powers γ^n, $n \in \mathbb{Z}$, have a contracting behavior on X: for example the positive powers γ^n, $n \geqslant 1$ push any compact set not containing the eigenline of maximal modulus of γ^{-1} inside a small neighborhood around the eigenline of maximal modulus of γ. Using irreducibility of the action, one then find a conjugate $c\gamma c^{-1}$ of γ such that $a := \gamma^n$ and $b = c\gamma^n c^{-1}$ exhibit the desired "ping-pong" behavior for all large enough n and thus generate a free subgroup. For details, see the original article [18] or e.g. [3].

It turns out that one can give a shorter proof of the corollary, which by-passes the proof of the existence of a free subgroup. This was observed by Shalom [16] and the argument, which unlike the proof of the Tits alternative does not require the theory of algebraic groups, is as follows.

Sketch of a direct proof of Corollary 0.2. Let us first assume that Γ is an unbounded subgroup of $\mathrm{GL}_n(k)$, for some local field k, which acts strongly irreducibly (i.e. it does not preserve any finite union of proper linear subspaces). If Γ is amenable, then it must preserve a probability measure on $\mathbb{P}(k^n)$. However recall:

Lemma 0.4. *(Furstenberg's Lemma) Suppose μ is a probability measure on the projective space $\mathbb{P}(k^n)$. Then the stabilizer of μ in $\mathrm{PGL}_n(k)$ is compact unless μ is degenerate in the sense that it is supported on a finite number of proper (projective) linear subspaces.*

For the proof of this lemma, see Zimmer's book [20] or try to prove it yourself. Clearly the stabilizer of a degenerate measure preserves a finite union of proper subspaces. This contradicts our assumption.

To complete the proof, it remains to see that if Γ is not virtually solvable, then we can always reduce to the case above. This follows from two claims.

Claim 1. A linear group is not virtually solvable if and only if it has a finite index subgroup which has a linear representation in a vector space of dimension at least 2 which is absolutely strongly irreducible (i.e. it preserves no finite union of proper vector subspaces defined over any field extension).

Claim 2. If a finitely generated subgroup Γ of $\mathrm{GL}_d(K)$ acts absolutely strongly irreducibly on K^d, $d \geqslant 2$, and K is a finitely generated field, then K embeds in a local field k in such a way that Γ is unbounded in $\mathrm{GL}_d(k)$. $\square$

Exercise. Prove Claim 1.

The proof of Claim 2 requires some basic algebra and number theory as proceeds as follows.

Exercise. Prove that if a subgroup of $\mathrm{GL}_d(\overline{K})$ acts irreducibly ($\overline{K}$=algebraic closure) and all of its elements have only 1 in their spectrum (i.e. are unipotents), then $d = 1$ (hint: use Burnside's theorem that the only subalgebra of $M_n(\overline{K})$ acting absolutely irreducibly is all of $M_n(\overline{K})$.)
Exercise. Show that a finitely generated field K contains only finitely many roots of unity and that if $x \in K$ is not a root of unity, they there is a local field k with absolute value $|\cdot|$ such that K embeds in k and $|x| \neq 1$ (hint: this is based on Kronecker's theorem that if a polynomial in $\mathbb{Z}[X]$ has all its roots within the unit disc, then all its roots are roots of unity; see [18, Lemma 4.1] for a full proof).
Exercise. Use the last two exercises to prove Claim 2.

II. Kazhdan's property (T)

Let us go back to general (countable) groups and introduce another spectral property of groups, namely Kazhdan's property (T). Our goal here is to give a very brief introduction. Many excellent references exist on property (T) starting with the 1989 Asterisque monograph by de la Harpe and Valette [?], the recent book by Bekka-de la Harpe-Valette for the classical theory; see also Shalom 2006 ICM talk for more recent developments.

Let π be a unitary representation of Γ on a Hilbert space $\mathcal{H}_\pi$. We say that π admits (a sequence of) *almost invariant vectors* if there is a sequence of unit vectors $v_n \in \mathcal{H}_\pi$ ($||v_n|| = 1$) such that $||\pi(\gamma)v_n - v_n||$ converges to 0 as n tends to $+\infty$ for every $\gamma \in \Gamma$.

Definition 0.5. *(Kazhdan's property (T)) A group Γ is said to have* Kazhdan's property (T) *if every unitary representation π admitting a sequence of almost invariant vectors admits a non-zero Γ-invariant vector.*

Groups with property (T) are sometimes also called *Kazhdan groups.*

A few simple remarks are in order following this definition:

- The definition resembles that of non-amenability, except that we are now considering all unitary representations of Γ and not just the left regular reprensentation $\ell^2(\Gamma)$ (given by $\lambda(\gamma)f(x) := f(\gamma^{-1}x)$). Indeed by Proposition **??**(3) above shows that a group is amenable if and only if the regular representation on $\ell^2(\Gamma)$ admits a sequence of almost invariant vectors.
- Property (T) is inherited by quotient groups of Γ (obvious from the definition).
- Finite groups have property (T) (simply average an almost invariant unit vector over the group).
- If Γ has property (T) and is amenable, then Γ is finite (indeed $\ell^2(\Gamma)$ has a non-zero invariant vector iff the constant function 1 is in $\ell^2(\Gamma)$ and this is iff Γ is finite).

A first important consequence[1] of property (T) is the following:

Proposition 0.6. *Every countable group with property (T) is finitely generated.*

Proof. Let S_n be an increasing family of finite subsets of Γ such that $\Gamma = \bigcup_n S_n$. Let $\Gamma_n := \langle S_n \rangle$ be the subgroup generated by S_n. We wish to show that $\Gamma_n = \Gamma$ for all large enough n. Consider the left action of Γ on the coset space Γ/Γ_n and the unitary representation π_n it induces on ℓ^2 functions on that coset space, $\ell^2(\Gamma/\Gamma_n)$. Let $\pi = \oplus_n \pi_n$ be the Hilbert direct sum of the $\ell^2(\Gamma/\Gamma_n)$ with the natural action of Γ on each factor. We claim that this unitary representation of Γ admits a sequence of almost invariant vectors. Indeed let v_n be the Dirac mass at $[\Gamma_n]$ in the coset space Γ/Γ_n. We view v_n as a (unit) vector in π. Clearly for every given $\gamma \in \Gamma$, if n is large enough γ belongs to Γ_n and hence preserves v_n. Hence $||\pi(\gamma)v_n - v_n||$ is equal to 0 for all large enough n and the $(v_n)_n$ form a family of almost invariant vectors. By Property (T), there is a non-zero invariant vector $\xi := \sum_n \xi_n$. The Γ-invariance of ξ is equivalent to the Γ-invariance of all $\xi_n \in \ell^2(\Gamma/\Gamma_n)$ simultaneously. However observe that if $\xi_n \neq 0$, then Γ/Γ_n must be finite (otherwise a non-zero constant function cannot be in ℓ^2). Since there must be some n such that $\xi_n \neq 0$, we conclude that some Γ_n has finite index in Γ. But Γ_n itself is finitely generated. It follows that Γ is finitely generated. $\square$

So let Γ have property (T), and let S be a finite generating set for Γ. Then from the very definition we observe that there must be some $\varepsilon = \varepsilon(S) > 0$ such that for every unitary representation π of Γ without non-zero Γ-invariant vectors, one has:

$$\max_{s \in S} ||\pi(s)v - v|| \geqslant \varepsilon ||v||,$$

for every vector $v \in \mathcal{H}_\pi$.

And conversely it is clear that if there is a finite subset S in Γ with the above property, then every unitary representation of Γ with almost invariant vectors has an invariant vector. Hence this is equivalent to Property (T).

[1]This was in fact the reason for its introduction by Kazhdan in 1967 (at age 21). He used it to prove that non-uniform lattices in (higher rank) semisimple Lie groups are finitely generated. Nowadays new proofs exist of this fact, which are purely geometric and give good bounds on the size of the generating sets, see Gelander's lecture notes from the PCMI summer school.

Definition 0.7. *(Kazhdan constant) The (optimal) number $\varepsilon(S) > 0$ above is called a Kazhdan constant for the finite set S.*

Another important property of Kazhdan groups is that they have finite abelianization:

Proposition 0.8. *Suppose Γ is a countable group with property (T). Then $\Gamma/[\Gamma,\Gamma]$ is finite.*

Proof. Indeed, $\Gamma/[\Gamma,\Gamma]$ is abelian hence amenable. It also has property (T), being a quotient of a group with property (T). Hence it is finite (see itemized remark above). $\square$

This implies in particular that the non-abelian free groups do not have property (T) although they are non-amenable. In fact Property (T) is a rather strong spectral property a group might have. I tend to think of it as a rather rare and special property a group might have (although in some models of random groups, almost every group has property (T)).

Exercise. Show that if Γ has a finite index subgroup with property (T), then it has property (T). And conversely, if Γ has property (T), then every finite index subgroup also has property (T).

In fact establishing Property (T) for any particular group is never a simple task. In his seminal paper in which he introduced Property (T) Kazhdan proved that Property (T) for simple Lie groups of rank[2] at least 2. Then he deduced (as in the above exercise) that Property (T) is inherited by all discrete subgroups of finite co-volume in the Lie group G (i.e. lattices).

Theorem 0.9. *(Kazhdan 1967) A lattice in a simple real Lie group of real rank at least 2 has property (T).*

There are several proofs of Kazhdan's result for Lie groups (see e.g. Zimmer's book [20] and Bekka-delaHarpe-Valette [2] for two slightly different proofs). They rely of proving a "relative property (T)" for the pair $(\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$. This relative property (T) means that every unitary representation of the larger group with almost invariant vectors admits a non-zero vector which is invariant under the smaller group. One proof of this relative property makes use of Furstenberg's lemma above (Lemma 0.4). The proof extends to simple groups defined over a local field with rank at least 2 (over this local field).

Using a more precise understanding of the irreducible unitary representations of simple real Lie groups of rank one Kostant was able to prove that the rank one groups $Sp(n,1)$ and F_4^{-20} have property (T). However the other rank one groups $SU(n,1)$ and $SO(n,1)$ (including $SL_2(\mathbb{R})$) do not have property (T).

[2]In fact he proved it for rank at least 3 by reducing the proof to $\mathrm{SL}_3(\mathbb{R})$ since every simple real Lie group of rank at least 3 contains a copy of $\mathrm{SL}_3(\mathbb{R})$, but it was quickly realized by others (treating the case of $Sp_4(\mathbb{R})$) that the argument extends to groups of rank 2 as well.

The discrete group $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$ and hence has property (T) by Kazhdan's theorem. Nowadays (following Burger and Shalom) they are more direct proofs that $\mathrm{SL}_n(\mathbb{Z})$ has property (T) (see the exercise sheet for Shalom's proof using bounded generation). Other examples of groups with property (T) include

Recently property (T) was established for $\mathrm{SL}_n(R)$, $n \geqslant 3$, where R is an arbitrary finitely generate commutative ring with unit, and even for $EL_n(R)$ for certain non-commutative rings R. For example:

Theorem 0.10. *(Ershov and Jaikin-Zapirain [**?**]) Let R be a (non-commutative) finitely generated ring with unit and $EL_n(R)$ be the subgroup of $n \times n$ matrices generated by the elementary matrix subgroups $Id_n + RE_{ij}$. If $n \geqslant 3$, then $EL_n(R)$ has property (T).*

In particular, if $\mathbb{Z}\langle x_1, \dots, x_k\rangle$ denotes the free associative algebra on k generators, $EL_n(\mathbb{Z}\langle x_1, \dots, x_k\rangle)$ has property (T) for all $k \geqslant 0$ and $n \geqslant 3$. As an other special case, the so-called *universal lattices* $EL_n(\mathbb{Z}[x_1, ..., x_k]) = \mathrm{SL}_n(\mathbb{Z}[x_1, ..., x_k])$, where $\mathbb{Z}[x_1, ..., x_k]$ is the ring of polynomials on k (commutative) indeterminates has property (T) when $n \geqslant 3$. This remarkable result extends earlier works of Kassabov, Nikolov and Shalom on various special cases. The Kazhdan constant in this case behave asymptotically as $\frac{1}{\sqrt{n+k}}$ for large n and k.

An important tool in some of these proofs (e.g. see Shalom ICM talk [17]) is the following characterization of property (T) in terms of affine actions of Hilbert spaces.

Theorem 0.11. *(Delorme-Guichardet) A group Γ has property (T) if and only if every action of Γ by affine isometries on a Hilbert space must have a global fixed point.*

See [**?**] or [2] for a proof. Kazhdan groups enjoy many other fixed point properties (e.g. Serre showed that they cannot act on trees without a global fixed point) and related rigidity properties (see e.g. the lectures by Dave Morris in this summer school).

Although the above class of examples of groups with property (T) all come from the world of linear groups, Kazhdan groups also arise geometrically, for example as hyperbolic groups through Gromov's random groups. For example the following holds:

Theorem 0.12. *In the density model of random groups, if the density is $< \frac{1}{2}$, then the random group is infinite and hyperbolic with overwhelming probability. If the density is $> \frac{1}{3}$, then the random group has property (T) with overwhelming probability.*

It is unknown whether $\frac{1}{3}$ is the right threshold for property (T). Below $\frac{1}{12}$ random groups have small cancellation $C'(1/6)$ and Ollivier and Wise proved that below $\frac{1}{6}$ they act freely and co-compactly on a $CAT(0)$ cube complex and are Haagerup, hence they do not have property (T).

For a proof of the above see Zuk [**?**], Ollivier [**?**], Gromov [**?**] and Ghys' Bourbaki talk [**?**]. In fact Zuk proved a similar result for a slightly different model of random groups (the so-called triangular model) and Ollivier sketches a reduction of the above to Zuk's theorem in [?]. The proof of this result is based on a geometric criterion for property (T) (due to Zuk, Ballmann-Zwiatkowski, originating in the work of Garland).

Let Γ be a group generated by a finite symmetric set S (with $e \notin S$). Let $L(S)$ be the finite graph whose vertices are the elements of S and an edge is drawn between two vertices s_1 and s_2 iff $s_1^{-1}s_2$ belongs to S. Suppose that $L(S)$ is connected (this is automatic if S is replaced say by $S \cup S^2 \setminus \{e\}$).

Theorem 0.13. *(local criterion for property (T)) Let Γ be a group generated by a finite symmetric set S (with $e \notin S$) such that the first non-zero eigenvalue of the Laplacian on the finite graph $L(S)$ is $> \frac{1}{2}$. Then Γ has property (T).*

For a short proof, see Gromov's random walks in random groups paper [?] and the end of Ghys' Bourbaki talk [?].

For certain groups of geometric origin, such as $Out(F_n)$ and the mapping class groups, determining whether they have property (T) are not can be very hard. For example it is not known whether $Out(F_n)$ has property (T) for $n \geqslant 4$ (even open for $Aut(F_n)$, not true for $n = 2, 3$ though). For the mapping class group, check the work of Andersen.

III. Uniformity issues in the Tits alternative, non-amenability and Kazhdan's property (T)

A well-known question of Gromov from [12] is whether the various invariants associated with an infinite group (such as the rate of exponential growth, the isoperimetric constant of a non-amenable group, the Kazhdan constant of a Kazhdan group, etc) can be made uniform over the generating set.

For example we say:

Definition 0.14. *Consider the family of all finite symmetric generating sets S of a given finitely generated group. Γ. We say that Γ*

- *has uniform exponential growth if $\exists \varepsilon > 0$ such that $\lim \frac{1}{n} \log |S^n| \geqslant \varepsilon$, for all S,*
- *is uniformly non-amenable if $\exists \varepsilon > 0$ such that $|\partial_S A| \geqslant \varepsilon |A|$ for S,*
- *has uniform property (T) if $\exists \varepsilon > 0$ such that $\max_S ||\pi(s)v - v|| \geqslant \varepsilon ||v||$ for all S and all unitary representations of Γ with no non-zero invariant vector.*
- *satisfies the uniform Tits alternative if $\exists N \in \mathbb{N} > 0$ such that S^N contains generators of a non-abelian free subgroup F_2.*

Note that there are some logical implications between these properties. For example if Γ satisfies the uniform Tits alternative, or if Γ (is infinite and) has uniform property (T), then Γ is uniformly non-amenable (exercise). Similarly if Γ is uniformly non-amenable, then Γ has uniform exponential growth.

Uniform exponential growth holds for linear groups of exponential growth (Eskin-Mozes-Oh [8]), for solvable groups of exponential growth (Osin), but fails for general groups as Wilson gave an example of a non-amenable group (even containing F_2) whose exponential growth is not uniform [19].

The uniform Tits alternative is known to hold for non-elementary hyperbolic groups (Koubi) and for non-virtually solvable linear group is known by work of Breuillard-Gelander [4]. In this case the uniformity is even stronger as one has:

Theorem 0.15. *(B. [6]) Given $d \in \mathbb{N}$, there is $N = N(d) \in \mathbb{N}$ such that for any field K and any finite symmetric set $S \subset \mathrm{GL}_d(K)$ one has S^N contains two generators of a non-abelian free subgroup F_2 unless $\langle S \rangle$ is virtually solvable.*

The uniformity in the field here requires some non-trivial number theory (see [5]). Of course this result implies that the rate of exponential growth is also bounded below by a positive constant depending only on the size of the matrix and not on the field. So the uniform exponential growth is also uniform in the field. However this is known to hold only for non-virtually solvable groups. It is an open question as to whether or not it also holds uniformly over all virtually solvable subgroups of $\mathrm{GL}_d(K)$ of exponential growth. In fact even the case of solvable subgroups of $\mathrm{GL}_2(\mathbb{C})$ is open. One can show however that if this is indeed the case, then this would imply the Lehmer conjecture from number theory [7], and that the analogous uniform Tits alternative for free semi-groups does not hold.

The above uniform Tits alternative has applications outside the world of infinite linear groups. It turns out that the uniformity in the field allows one to transfer information from the infinite world to the finite world (we'll see more of that in the remainder of this course). For example the following can be derived from Theorem 0.15

Corollary 0.16. *There is $N = N(d) \in \mathbb{N}$ and $\varepsilon = \varepsilon(d) > 0$ such that if S is a generating subset of $\mathrm{SL}_d(\mathbb{F}_p)$ (p arbitrary prime number), then S^N contains two elements a, b which generate $\mathrm{SL}_d(\mathbb{F}_p)$ and have no relation of length $\leqslant (\log p)^\varepsilon$. In other words the Cayley graph $\mathcal{G}(\mathrm{SL}_d(\mathbb{F}_p), \{a^{\pm 1}, b^{\pm 1}\})$ has girth at least $(\log p)^\varepsilon$.*

It is an open question whether one can take $\varepsilon = 1$ in the above result.

Uniform property (T) is even more mysterious. Examples where constructed by Osin and Sonkin [14]. Lattices in semisimple Lie groups with property (T) do not have uniform property (T) in general (examples were constructed by Gelander and Zuk). But it is an open problem to determine whether $\mathrm{SL}_n(\mathbb{Z})$ has uniform property (T) for $n \geqslant 3$.

1. Lecture 3: Property (τ) and expanders

To be continued...

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