EXERCISE SHEET 2

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1. Some facts on expanders

1) (Schreier graphs as expanders) Suppose G is a finite group with symmetric generating set S and $\mathcal{C}(G,S)$ is its Cayley graph. Suppose G acts transitively on a set X and let $\mathcal{S}(X,S)$ be the Schreier graph of this action, namely the graph with vertex set X and edges given by $x \simeq y$ iff there is $s \in S$ s.t. $x = s \cdot y$.

Show that $\lambda_2(\mathcal{S}(X,S)) \leqslant \lambda_2(\mathcal{C}(G,S))$. In particular the quotient Schreier graphs obtained from a family of expanders Cayley graphs are expander graphs. (hint: show that every eigenvalue of the Schreier graph is an eigenvalue of the Cayley graph).

Deduce that if Γ is a finitely generated group which has property (τ) with respect to a family of finite index normal subgroups $(\Gamma_i)_i$ which is such that every finite index subgroup of Γ contains some Γ_i, then Γ has property (τ). In particular the Selberg property for an arithmetic lattice combined with the congruence subgroup property imply property (τ).

2. Some basics on approximate groups

3) (Ruzsa's triangle inequality) Given two finite sets A, B in an ambient group G, we set

$$d(A,B) := \log \frac{|AB^{-1}|}{|A||B|},$$

a quantity called the *Ruzsa distance* between the sets A and B.

1) Show the triangle inequality: given any three finite sets A, B, C in G.

$$d(A,C) \leqslant d(A,B) + d(B,C),$$

[hint: consider the map $(b,x) \mapsto (a_x^{-1}b, b^{-1}c_x)$, where $a_x^{-1}c_x$ is a representation of $x \in A^{-1}C$.]

2) Deduce that if A is a finite subset of G such that $|AAA| \leqslant K|A|$ for some real number $K \geqslant 0$, then for every integer $n \geqslant 3$ we have $|A^n| \leqslant K^{2n-5}|A|$.

4) (Sets of small doubling)

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0) Let A be a finite subset of a group G such that $|AA| = |A|$. Show that $A = aH$, where H is a finite group and a normalizes H.

Now suppose we only know that

$$|AA| \leqslant K|A|,$$

for some $K \geqslant 1$.

1) Show that $|A^{-1}A| \leqslant K^2|A|$.

2) Show that if K is close enough to 1, then the only such sets A must be contained in a coset of a subgroup H of G of size at most $2|A|$ (hint: show that $A^{-1}A$ is a group).

3) Push the argument to prove that $|AA| < \frac{3}{2}|A|$ if and only if there is a subgroup H of G and a in the normalizer of H in G such that $A \subset aH$ and $|A| > \frac{2}{3}|H|$ (hint: show first that $A^{-1}A = AA^{-1}$).

References

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