# EXERCISES FOR THE PCMI SUMMER SCHOOL

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## 1. Amenability, paradoxical decompositions and Tarski numbers

In this exercise, we prove yet another characterization of amenability, which is due to Tarski [7, 4] and states that a group is non-amenable if and only if it is paradoxical.

Let $\Gamma$ be a group acting on a set $X$. This $\Gamma$-action is said to be $N$*-paradoxical* if one can partition $X$ into $n + m \leqslant N$ disjoint pieces

$$X = A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m$$

in such a way that there are elements $a_1, \ldots, a_n \in \Gamma$ and $b_1, \ldots, b_m \in \Gamma$ such that

$$X = \bigcup_{i=1}^{n} a_i A_i \text{ and } \bigcup_{j=1}^{m} b_j B_j$$

We say that $\Gamma$ is paradoxical if it is $N$-paradoxical for some finite $N \in \mathbb{N}$ for the action of $\Gamma$ on itself by left translations.

**1)** Prove that the non-abelian free group $F_2$ and in fact any group $\Gamma$ containing the free group $F_2$ is 4-paradoxical.

**2)** Suppose that $\Gamma$ is a 4-paradoxical group and $\Gamma = A_1 \cup A_2 \cup B_1 \cup B_2$ is a paradoxical decomposition as defined above. Show that $\Gamma$ *plays ping-pong* on itself, where the ping-pong players are $a := a_1^{-1}a_2$ and $b := b_1^{-1}b_2$ .Deduce that $\Gamma$ contains a non-abelian free subgroup $F_2$.

**3)** Define the *Tarski number* $\mathcal{T}(\Gamma)$ of a group $\Gamma$ to be the smallest integer $N$ if it exists such that $\Gamma$ is $N$-paradoxical. By the above $\mathcal{T}(\Gamma) = 4$ if and only if $\Gamma$ contains $F_2$. Show that if $\Gamma$ is amenable, then $\mathcal{T}(\Gamma) = +\infty$.

**4)** Suppose that $\Gamma$ is finitely generated with symmetric generating set $S$ and is endowed with the corresponding word metric $d$ (i.e. $d(x, y) := \inf\{n \in \mathbb{N}, x^{-1}y \in S^n\}$). Show that $\mathcal{T}(\Gamma)$ is finite if and only if there exists a surjective 2-to-1 mapping $\phi : \Gamma \to \Gamma$ with the property that $\sup_{\gamma \in \Gamma} d(\gamma, \phi(\gamma)) < +\infty$.

**5)** Given $k \in \mathbb{N}$, let $\mathcal{G}_k$ be the bi-partite graph obtained by taking two copies $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ as the left and right vertices respectively and by placing an edge between $\gamma \in \Gamma_1$

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and $\gamma' \in \Gamma_2$ if and only if $d(\gamma, \gamma') \leqslant k$ in the word metric of $\Gamma$. Show that if there is some finite $k \in \mathbb{N}$ such that $\mathcal{G}_k$ admits a $(2,1)$ perfect matching[1], then $\Gamma$ is paradoxical.

**6)** Prove the following version of *Hall's marriage lemma* for infinite bi-partite graphs. Let $k$ be a positive integer (we will need the result for $k = 2$ only). Suppose $\mathcal{B}$ is a bi-partite graph whose set of left vertices is countable infinite as is the set of right vertices. Suppose that for every finite subset of left vertices $L$, the number of right vertices connected to some vertex in $L$ has size at least $k|L|$, while for every finite subset $R$ of right vertices, the number of left vertices connected to some vertex in $R$ has size at least $|R|$. Show that $\mathcal{B}$ admits a $(k,1)$ perfect matching. [Hint: first treat the case $k = 1$, then reduce to this case.]

**7)** Using 6) that if $\Gamma$ is a non-amenable finite generated group, then there is $k \geqslant 1$ such that $\mathcal{G}_k$ has a $(2,1)$ perfect matching.

**8)** Conclude the proof of Tarski's theorem for arbitrary (not necessarily finitely generated) groups.

## 2. Kazhdan's property $(T)$ for $\mathrm{SL}_n(\mathbb{Z})$ via bounded generation

The goal of this exercise is to show how bounded generation can be useful to establish property $(T)$ and to provide a proof (due to Shalom [6]) that $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 3$, has property $(T)$ along these lines.

**I. Preliminaries for general $G$.**

**1)** Show that property $(T)$ for a finitely generated group $G$ is equivalent to the following. Given a finite generating set $S$, there is $C > 0$ such that for any $\varepsilon > 0$ and any unitary representation $(\pi, \mathcal{H}_\pi)$ of $G$, if $v \in \mathcal{H}_\pi$ satisfies $||\pi(s)v - v|| \leqslant \varepsilon$ for all $s \in S$, then there is an $G$-invariant vector $w \in \mathcal{H}_\pi$ such that $||v - w|| \leqslant C\varepsilon$.

**2)** Let $\varepsilon \leqslant 1$. Suppose $(\pi, \mathcal{H}_\pi)$ is a unitary representation of $G$ and $v$ is a unit vector in $\mathcal{H}_\pi$ such that $\sup_{g \in G} ||\pi(g)v - v|| \leqslant \varepsilon$. Show that $G$ admits a non-zero invariant vector $w$ with $||v - w|| < \varepsilon$. (Hint: use the circumcenter).

**3)** Suppose $G$ admits finitely (or compactly) generated subgroups $H_1, \ldots, H_n$ with property $(T)$ such that $G = H_1 \ldots H_n$ in the sense that any element of $G$ can be written as a product $h_1 \cdot \ldots \cdot h_n$ with $h_i \in H_i$ (i.e. $G$ is *boundedly generated by the $H_i$'s*). Show that $G$ has property $(T)$.

Deduce from this that in order to prove that $\mathrm{SL}_n(\mathbb{R})$ has property $(T)$ it is enough to prove that $\mathrm{SL}_3(\mathbb{R})$ has property $(T)$.

**4)** A pair $(G, H)$ of groups, where $H$ is a subgroup of $G$, is said to have *relative property* $(T)$ if any unitary representation of $G$ with almost invariant vectors admits an

[1] By definition this is a subset of edges of $\mathcal{G}_k$ such that the induced bi-partite graph has the property that every vertex on the left hand side is connected to exactly two vertices on the right hand side, while every vertex on the right hand side is connected to exactly one vertex on the left hand side.

$H$-invariant vector. Show that if $G$ is boundedly generated by subgroups $H_1, \dots, H_n$ and each $H_i$ is normalized by a finitely (or compactly) generated subgroup $L_i \leqslant G$ such that $(L_i, H_i)$ has relative property $(T)$, then $G$ has property $(T)$.

**II. Bounded generation for** $\mathrm{SL}_n(\mathbb{Z})$**.**

**1)** Prove that $\mathrm{SL}_n(\mathbb{R})$, $n \geqslant 2$, is boundedly generated by its *elementary subgroups*, namely the subgroups $H_{ij}$ of the form $Id_n + \mathbb{R}E_{ij}$, where $E_{ij}$ is the elementary $n \times n$ matrix all of whose entries are 0 except the $ij$ entry.

**2)** Using elementary operations on rows and columns, show that in order to prove that $\mathrm{SL}_n(\mathbb{Z})$, $n \geqslant 3$, is boundedly generated by its elementary subgroups $Id + \mathbb{Z}E_{ij}$, it is enough to do this for $n = 3$. (hint: reduce to the case when the first $n-1$ entries of the bottom row of a given matrix in $\mathrm{SL}_n(\mathbb{Z})$ are relatively prime).

We won't do the $n = 3$ case here. Anybody interested in advised to look at [1, Lemma 4.1.6] or the original article [3].

**III. Relative property** $(T)$ **for** $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$**.**

We want to prove that if $(\pi, \mathcal{H})$ is a unitary representation of $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ with almost invariant vectors, then there is a $\mathbb{Z}^2$ invariant vector. We follow an argument due to M. Burger [2].

The restriction $\pi_{|\mathbb{Z}^2}$ is a unitary representation of $\mathbb{Z}^2$. The dual of $\mathbb{Z}^2$ is the torus $T := \widehat{\mathbb{Z}^2} = (\mathbb{R}/\mathbb{Z})^2$. Recall that according to the spectral theorem, there exists a resolution of identity $E : T \to \mathcal{B}(\mathcal{H})$, which assigns to every Borel set $\Omega \subset T$ a self-adjoint projection $E(\Omega) : \mathcal{H} \to \mathcal{H}$, such that

a) $E(\varnothing) = 0$, $E(T) = Id$, $E(A \cap B) = E(A)E(B)$, and $E(A \cup B) = E(A) + E(B)$ if $A \cap B = \varnothing$.

b) for every $v, w \in \mathcal{H}$, $\Omega \mapsto (E(\Omega)v, w)$ is complex measure and for all $\xi \in \mathbb{Z}^2$

$$(\pi(\xi)v, w) = \int_T e^{2i\pi\xi\cdot\omega}(E(d\omega)v, w)$$

The idea of the proof is to study the probability measures $\mu_v(\Omega) := (E(\Omega)v, v)$ when $v$ is an almost invariant vector and to show, using the action of $\mathrm{SL}_2(\mathbb{Z})$, that they must charge $\{0\}$, implying that $\pi_{|\mathbb{Z}^2}$ has invariant vectors. Now come the details.

**1)** Verify that a) and b) imply that given $\omega \in T$, l'image $ImE(\{\omega\})$ is the joint eigenspace of $\mathbb{Z}^2$ with eigenvalue $e^{2i\pi\omega\cdot\xi}$ (i.e. $\forall \xi \in \mathbb{Z}^2$, $\pi(\xi)v = e^{2i\pi\omega\cdot\xi}v$ iff $v \in ImE(\{\omega\})$). In particular $E(\{0\})$ is the orthogonal projection to the invariant vectors.

**2)** Let $v \in \mathcal{H}$ be a unit vector and $\mu_v$ be the probability measure on $T$ given by $\Omega \mapsto (E(\Omega)v, v)$. Show that $|\mu_v(B) - \mu_w(B)| \leqslant 2||v - w||$ for every Borel set $B \subset T$ and all $v, w \in \mathcal{H}$. Also check that $g_*\mu_v = \mu_{\pi(g)v}$, where $g \in \mathrm{SL}_2(\mathbb{Z})$ acts on $T$ in the natural way and $g_*\mu(\Omega) := \mu(g^{-1}\Omega)$.

**3)** Given a sequence of almost invariant unit vectors $(v_k)_k$ for $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, show that $\mu_{v_k}$ converges weakly to the Dirac mass at $0 \in T$.

**4)** If $\pi$ has no $\mathbb{Z}^2$ invariant vectors, show that $\mu_v(\{0\}) = 0$ for every $v$.

**5)** Let $a := Id+2E_{12}$ and $b := Id+2E_{21}$ be two elementary matrices in $\mathrm{SL}_2(\mathbb{Z})$. Show the following lemma (which is one way to show that $\mathrm{SL}_2(\mathbb{Z})$ has no invariant measure on the projective line $\mathbb{P}(\mathbb{R}^2)$ and hence is non-amenable). There is $\varepsilon_0 > 0$ such that for every probability measure $\mu$ on $\mathbb{R}^2 \setminus \{0\}$, there is a Borel subset $Y \subset \mathbb{R}^2 \setminus \{0\}$ such that $|\mu(gY) - \mu(Y)| \geqslant \varepsilon_0$ for some $g \in \{a^{\pm 1}, b^{\pm 1}\}$.

**6)** Conclude.

**IV. Property** $(T)$ **for** $\mathrm{SL}_n(\mathbb{Z})$**.** Show finally that $\mathrm{SL}_n(\mathbb{Z})$ has property $(T)$ using bounded generation of the elementary subgroups $Id_n + \mathbb{Z}E_{pq}$ and relative property $(T)$ for the pair $(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$. (hint: set the $H_i$'s to be the subgroups generated by two distinct elementary subgroups and find subgroups $L_i$ in $\mathrm{SL}_n(\mathbb{Z})$ such that $H_i \simeq \mathbb{Z}^2$ is normal in $L_i \simeq \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, then use $I$.4. to conclude.)

Does this work for $\mathrm{SL}_n(\mathbb{R})$ ?

## 3. Harmonic functions and property $(T)$

One of the amazing things about property $(T)$ is that it can be used to prove theorems that at first sight seem far removed from any question involving spectral gaps or unitary representations. The most outstanding example of this is Margulis' famed proof of the Normal Subgroup Theorem (which states that a normal subgroup in a higher rank lattice is either finite or of finite index). Margulis' proof proceeds by showing that any quotient of the lattice by a non-central subgroup is both amenable and has property $(T)$ hence is finite.

Another such example is the following fact, which is a key step in Kleiner's proof of Gromov's polynomial growth theorem ([5] and the references therein):

**Theorem[2]:** Any finitely generated infinite group admits a non-constant Lipschitz harmonic function.

Let $\Gamma$ be a group generated by a finite symmetric set $S = \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$. A function on $\Gamma$ is said to be harmonic if for all $x \in \Gamma$

$$f(x) = \frac{1}{|S|} \sum_{s \in S} f(xs)$$

We say it is Lipschitz if $|f(x) - f(y)| \leqslant C d(x, y)$ for some $C > 0$ and all $x, y \in \Gamma$, where $d(x, y)$ is the word metric induced by $S$ on $\Gamma$.

[2]This result is almost a counter-example to the somewhat provocative assertion I once heard according to which there is no property that is both non-trivial and holds for all finitely generated groups.

The goal of this exercise will be to prove this theorem. The proof splits in two parts: first we treat the case when $\Gamma$ is non-amenable. Then the case when $\Gamma$ does not have property $(T)$. A finitely generated group which does not fall into one of these two categories must be finite, hence the result.

**0)** First prove that on a finite group, every harmonic function is constant (hint: maximum principle).

**I. The non-amenable case.**

**1)** (Bogolyubov) Let $\mu$ be the symmetric and finitely supported measure $\mu := \frac{1}{|S|}\sum_{s\in S}\delta_s$ on $\Gamma$. Show that every action of $\Gamma$ by homeomorphisms on a compact space $X$ admits a stationnary measure, namely a Borel probability measure $\nu$ on $X$ such that $\mu * \nu = \nu$ (i.e. $\frac{1}{|S|}\sum_{s\in S}\nu(sA) = \nu(A)$ for every Borel subset $A \subset X$).

**2)** (Building bounded harmonic functions) Let $\nu$ is a stationary measure for $\mu$ on a compact $\Gamma$-space $X$. Show that for every bounded continuous function $f$ on $X$ the function $\phi_f : \gamma \mapsto \int_X f(\gamma \cdot x)d\nu(x)$ is harmonic and bounded on $\Gamma$.

**3)** Use 1) and 2) to prove that if $\Gamma$ is non-amenable, then $\Gamma$ admits a non-constant bounded harmonic function.

**II. Negating Property $(T)$.**

Recall that according to the Delorme-Guichardet theorem, $\Gamma$ has property $(T)$ if and only if every affine isometric action of $\Gamma$ on a Hilbert space has a global fixed point (property $(FH)$). We will need the following stronger fact: if $\Gamma$ does not have property $(T)$, then there is an affine isometric action on some Hilbert space $\mathcal{H}$ such that the $\ell^2$-displacement function

$$D_S(x) := \big(\sum_{s\in S} d(x, s\cdot x)^2\big)^{\frac{1}{2}}$$

is everywhere positive and attains its minimum at some point $x_0 \in \mathcal{H}$. Here $d(x,y)^2 = ||x-y||^2$ is the (square of the) distance in the Hilbert space $\mathcal{H}$ and $\gamma \cdot x$ denotes the affine action of $\Gamma$ in $\mathcal{H}$.

The proof we give below of this strengthening of $(FH) \Rightarrow (T)$ uses *ultralimits*, which are an extremely useful tool in all sorts of contexts when one wants to make uniform a seemingly non-uniform statement.

We briefly recall the construction of an ultralimit of metric spaces. We refer the reader to Misha Kapovich notes from this year's Park City summer school for more details. A non-principal ultrafilter $\omega$ is a set of subsets of $\mathbb{N}$ such that a) $A \in \omega$ and $A \subset B \Rightarrow B \in \omega$, b) if $A, B \in \omega$, then $A \cap B \in \omega$, c) for every $A \subset \mathbb{N}$, either $A$ or $A^c$ belongs to $\omega$, and d) no singleton belongs to $\omega$ (check that this is equivalent to the definition in terms of $\{0,1\}$-valued finitely additive measures given in Misha's notes). The existence of a non-principal ultrafilter is guaranteed by Zorn's lemma.

Given a sequence of pointed metric spaces $(X_n, d_n, x_n)$ recall that their *ultralimit* along $\omega$ is defined as the set of equivalence classes of sequences $(y_n)_n$ such that $\forall n, y_n \in X_n$ and $\sup_n d_n(y_n, x_n) < +\infty$, where $(y_n)$ and $(y'_n)$ are equivalent if $\lim_\omega d_n(y_n, y'_n) = 0$.

Let now $\mathcal{H}$ be a Hilbert space endowed with an affine isometric action of $\Gamma$. Let $d$ be the Euclidean distance on $\mathcal{H}$. Given a sequence of scalars $\lambda_n > 0$, and a sequence of base points $x_n \in \mathcal{H}_n$, we can form the ultralimit of the sequence of pointed metric spaces $(\mathcal{H}, \lambda_n d, x_n)$, say $(\mathcal{H}_\omega, d_\omega, x_\omega) = \lim_\omega(\mathcal{H}, \lambda_n d, x_n)$.

**1)** Show that $(\mathcal{H}_\omega, d_\omega, x_\omega)$ is again a Hilbert space endowed with an affine isometric action of $\Gamma$.

**2)** Let $\mathcal{H}$ be a Hilbert space endowed with an affine isometric action of $\Gamma$ admitting no global fixed point. Show that for every $n \in \mathbb{N}$, there is $x_n \in \mathcal{H}$ such that for every $y$ in the ball of radius $nD_S(x_n)$ one has

$$D_S(y) \geqslant (1 - \frac{1}{n})D_S(x_n).$$

**3)** Prove the aforementioned strengthening of $(FH) \Rightarrow (T)$ using an ultralimit of a renormalised sequence of pointed Hilbert spaces contradicting the uniformity.

**4)** Show that if $x_0$ realizes the minimum of $D_S(x)$ and $D_S(x_0) > 0$, then for every $v \in \mathcal{H}$ the map $\gamma \mapsto Re(\gamma \cdot x_0, v)$ (Re = real part) is harmonic and Lipschitz (hint: differentiate $D_S(x)^2$).

**5)** Conclude the proof of the theorem from the introduction. Can one make the harmonic function unbounded ?

## References


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