Some arithmetic groups that do not act on the circle

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Lecture 4
Intro to bounded cohomology
(used to prove actions have a fixed point)

Recall: group cohomology $H^*(\Gamma;\mathbb{R})$

- *cochain* $c\colon \Gamma^n \to \mathbb{R}$
- *coboundary* $\delta\colon C^n(\Gamma) \to C^{n+1}(\Gamma)$
- $H^n(\Gamma;\mathbb{R}) = \dfrac{Z^n(\Gamma;\mathbb{R})}{B^n(\Gamma;\mathbb{R})} = \dfrac{\ker \delta_n}{\operatorname{Im} \delta_{n-1}}$

Definition (bounded cohomology)

$H_b^*(\Gamma;\mathbb{R})$: require all cochains to be *bdd* funcs on Γ^n.

Example

- $H^0(\Gamma;\mathbb{R}) = \mathbb{R} = H_b^0(\Gamma;\mathbb{R})$.
- $H^1(\Gamma;\mathbb{R}) = \{$ homomorphisms $c\colon \Gamma \to \mathbb{R}\}$
- $H_b^1(\Gamma;\mathbb{R}) = \{$ *bounded* homos $c\colon \Gamma \to \mathbb{R}\} = \{0\}$.

Interested in $H_b^2(\Gamma)$ – applies to actions on the circle.

Example

Spse Γ acts on circle. I.e., $\Gamma \subset \mathrm{Homeo}_+(\mathbb{R}/\mathbb{Z})$.

Each $g \in \Gamma$ lifts to $\widetilde{g} \in \mathrm{Homeo}_+(\mathbb{R})$.

Not unique: $\widehat{g}(t) = \widetilde{g}(t) + n, \quad \exists n \in \mathbb{Z}$.

Can choose $\widetilde{g}(0) \in [0,1)$.

Let $c(g,h) = \widetilde{g}(\widetilde{h}(0)) - \widetilde{gh}(0) \in \mathbb{Z}$.

Exercise

- *c is a 2-cocycle:*
  $c(h,k) - c(gh,k) + c(g,hk) - c(g,h) = 0$
- $c(g,h) \in \{0,1\}$.

So $[c] \in H_b^2(\Gamma;\mathbb{Z})$. The bdd Euler class of the action.

Well defined: independent of basepoint "0", etc.

Bounded Euler class $c(g,h) = \widetilde{g}(\widetilde{h}(0)) - \widetilde{gh}(0)$

Proposition (Ghys)

$[c] = 0$ *in* $H_b^2(\Gamma;\mathbb{Z}) \iff \Gamma$ *has a fixed point in* S^1.

Proof.

($\Leftarrow$) Wolog fixed point is $\overline{0}$.

Then $\widetilde{g}(0) = 0$, so $c(g,h) = 0$ for all g,h.

($\Rightarrow$) $c(g,h) = \varphi(gh) - \varphi(g) - \varphi(h)$, $\exists$ *bdd* $\varphi\colon \Gamma \to \mathbb{Z}$.

Let $\widehat{g}(x) = \widetilde{g}(x) + \varphi(g)$, so

- $\widehat{g}\,\widehat{h} = \widehat{gh}$, so $\widehat{\Gamma}$ is a lift of Γ to $\mathrm{Homeo}_+(\mathbb{R})$, and
- $|\widehat{g}(0)| \le |\widetilde{g}(0)| + |\varphi(g)| \le 1 + \|\varphi\|_\infty$.

$\widehat{\Gamma}$-orbit of 0 is bdd subset of $\mathbb{R}$, so has a supremum, which is fixed pt of $\widehat{\Gamma}$; img in S^1 is fixed pt of Γ. □

Proposition (Ghys)

$[c] = 0$ *in* $H_b^2(\Gamma;\mathbb{Z}) \iff \Gamma$ *has a fixed point in* S^1.

Corollary

$H_b^2(\Gamma;\mathbb{Z}) = 0 \Rightarrow$ *every action of* Γ *on* S^1 *has fixed pt.*

Exercise

$H_b^2(\Gamma;\mathbb{R}) = 0$, $H^1(\Gamma;\mathbb{R}) = 0$, Γ *is finitely generated*

$\implies$ *every action of* Γ *on* S^1 *has a finite orbit.*

Theorem (Burger-Monod)

Comparison map $H_b^2(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ *is injective if* Γ *is large arith group.*

Corollary (Ghys, Burger-Monod)

$\Gamma =$ *large arith group* *and* $H^2(\Gamma;\mathbb{R}) = 0$

$\implies$ *every action of* Γ *on* S^1 *has a finite orbit.*

Theorem (Burger-Monod)

Comparison map $H_b^2(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ *is injective if* Γ *is large arith group.*

Kernel of the comparison map

Let $c \in Z_b^2(\Gamma;\mathbb{R})$. Assume $[c] = 0$ in $H^2(\Gamma;\mathbb{R})$.

I.e., $c = \delta\alpha, \quad \exists \alpha \in C^1(\Gamma)$. So

$|\alpha(gh) - \alpha(g) - \alpha(h)| = |\delta\alpha(g,h)| \le \|c\|_\infty$ is bdd.

α is *almost* a homo — a *quasimorphism.*

Exercise

Kernel of $H_b^2(\Gamma) \to H^2(\Gamma)$ *is* $\dfrac{\mathrm{Quasimorphisms}(\Gamma,\mathbb{R})}{\mathrm{NearHom}(\Gamma,\mathbb{R})}$.

$\mathrm{NearHom}(\Gamma,\mathbb{R}) = \{\, \alpha\colon \Gamma \to \mathbb{R} \mid$ *bdd dist from homo* $\}$

Exercise

Kernel of $H_b^2(\Gamma) \to H^2(\Gamma)$ *is* $\dfrac{\text{Quasimorphisms}(\Gamma,\mathbb{R})}{\text{NearHom}(\Gamma,\mathbb{R})}$.

Example: $H_b^2(F_2)$ is infinite-dimensional.

Proof. Construct lots of quasimorphisms (not homos).

Homo $\varphi_a(x)$ = the (signed) # occurrences of a in x.

E.g., $\varphi_a(a^2ba^3b^{-3}a^{-7}b^2) = 2+3-7 = -2$.

Every homo $F_2 \to \mathbb{R}$ is a linear comb of φ_a and φ_b.

$\varphi_{ab}(x)$ = # occurrences of ab in x (reduced)

E.g., $\varphi_{ab}(a^2ba^3b^{-3}a^{-7}b^2) = 1-1 = 0$.

Exercise: 1) φ_w is a quasimorphism, $\forall$ reduced w.

2) φ_{a^k} is not within bdd distance of lin span of $\{\varphi_b, \varphi_a, \varphi_{a^{k+1}}, \varphi_{a^{k+2}}, \varphi_{a^{k+3}}, \ldots\}$.

Example: homomorphism $\varphi\colon \Gamma \to \mathbb{R}$

$\implies \{\, g \in \Gamma \mid \varphi(g) > 0 \,\}$ is normal semigroup.

Exercise: Spse $\varphi\colon \Gamma \to \mathbb{R}$ unbdd quasimorphism.

Stabilize: let $\overline{\varphi}(g) = \lim \varphi(g^n)/n$.

- $\overline{\varphi}$ is unbounded quasimorphism.
- $\overline{\varphi}(h^{-1}gh) = \overline{\varphi}(g)$.
- $\{\, g \in \Gamma \mid \overline{\varphi}(g) > C \,\}$ is normal semigroup.

Open Problem. For $\Gamma = \mathrm{SL}(3,\mathbb{Z})$:

- Every normal semigroup in Γ is a subgroup.
- $\forall g \in \Gamma$, e is a product of conjugates of g.
- $\nexists$ (nonempty) *bi*-invariant *partial* order on Γ.

All are equivalent. (**$100** for solution)

Exercises

1) $H_b^2(\Gamma;\mathbb{R}) = 0$, $H^1(\Gamma;\mathbb{R}) = 0$, Γ is finitely gen'd
$\implies$ every action of Γ on S^1 has a finite orbit.
[*Hint:* Short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$ yields long exact sequence $H_b^1(\Gamma;\mathbb{T}) \xrightarrow{\delta} H_b^2(\Gamma;\mathbb{Z}) \to H_b^2(\Gamma;\mathbb{R})$.]
2) Every quasimorphism is bounded on the set of commutators $\{x^{-1}y^{-1}xy\}$.
3) $\mathrm{SL}(3,\mathbb{Z})$ has no unbounded quasimorphisms.
[*Hint:* Use the fact that it is boundedly gen'd by elementary mats.]

Further reading

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