Some arithmetic groups that do not act on the circle

Dave Witte Morris

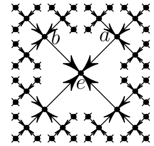
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Lecture 3
What is an amenable group?
(used to prove actions have a fixed point)

Amenability: fundamental notion in group theory.
Definition: dozens of choices (all equivalent).

Example

Free group $F_2 = \langle a, b \rangle$. Every el't starts with \$1:
$f_0(g) = 1, \quad \forall g \in F_2$.
Everyone passes their dollar to the person next to them who is closer to the identity:
$f_1(g) = \$3$ (except $f_1(e) = \$5$).



Everyone ≥\$2, & money only moved bdd distance.

Terminology

This is a *Ponzi scheme* on F_2.

Example

∃ *Ponzi scheme* on F_2:
Everyone ≥\$2, & money only moved bdd distance.

Exercise

On $\mathbb{Z}^n$, ∄ Ponzi scheme.
(*∃ Ponzi scheme ⟹ exponential growth.*)

Solvable grps of exp'l growth do *not* have a Ponzi:

Theorem (Gromov)

∄ Ponzi scheme on G ⟺ G is "amenable".

Corollary

Amenability is a geometric notion (*inv't under quasi-isom*).

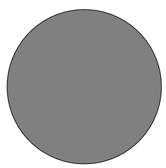
What is amenable really?

Answer

G is amenable ⟺ G has *almost-invariant* subsets.

Example

G = abelian group (f.g.) = $\mathbb{Z}^2 = \langle a, b \rangle$.
G acts on itself by left translation.
F = G-inv't subset of G, $\left(\substack{aF = F,\ bF = F, \\ \text{nonempty}}\right)$
⟹ F is infinite.



| ∄ finite, invariant subset. |

F = big ball ⟹ F is 99.99% invariant ("almost inv't"):
$\#(F \cap aF) > (1 - \epsilon)\,\#F$

Definition

F is *almost invariant* (F is a "Følner set"):
$\#(F \cap aF) > (1 - \epsilon)\#F \quad \forall a \in S$

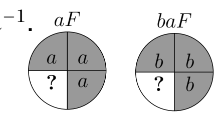
Definition

G amenable ⟺ G has almost-inv't finite subsets
($\forall$ finite S, $\forall \epsilon > 0$)

Exercise

Free group F_2 is not amenable.

Idea. $\frac{3}{4}$ of F does not start with a^{-1}.
⟹ $\frac{3}{4}$ of aF starts with a.
⟹ $\frac{3}{4}$ of baF starts with b.
$aF \approx F \approx baF$ ⟹ $\approx \frac{3}{4}$ of F starts with a and b. →←



Proposition

G amen ⟺ every bdd func on G has an avg value.
I.e., ∃ A: $\ell^\infty(G) \to \mathbb{R}$, s.t.

- $A(1) = 1$,
- $A(a\varphi + b\psi) = a\,A(\varphi) + b\,A(\psi)$,
- $A(\geq 0) \geq 0$,
- $A(\varphi^g) = A(\varphi)$. (*translation invariant*)

Proof.

Choose sequence of almost-inv't sets F_n ($\epsilon = 1/n$).
Let $A_n(\varphi) = \frac{1}{\#F_n} \sum_{x \in F_n} \varphi(x)$.
Pass to subsequence, so $A_n(\varphi) \to A(\varphi)$.
Can make a consistent choice of $A(\varphi)$ for all φ.
[Ultrafilter, Hahn-Banach, Zorn's Lemma, Tychonoff, Axiom of Choice] □

Bounded cohomology

Define group cohomology as usual, except that all cochains are assumed to be bounded functions.

Theorem (B. E. Johnson**)**

G amenable
⟺ $H^n_{\text{bdd}}(G; V) = 0$, *∀ G-module V* $\left(\substack{\text{such that } V \text{ is} \\ \text{a Banach space}}\right)$.

Proof of (⟹). If G is finite, and $|G|$ is invertible, one proves $H^n(G;V) = 0$ by averaging:
$\overline{\alpha}(g_1, \ldots, g_n) = \frac{1}{|G|} \sum_{g \in G} \alpha(g, g_1, \ldots, g_n)$.
Since G is amenable, we can do exactly this kind of averaging for any *bounded* cocycle.

Proposition

G amen ⟺ every bdd func on G has an avg value.

Average vals of characteristic funcs of subsets of G:

Corollary (von Neumann's original definition**)**

G amen ⟺ ∃ finitely additive probability measure.

Corollary (⟺)

- *G amenable,*
- *G acts on compact metric space X* (*by homeos*)

⟹ *every continuous function on X has an avg val*
⟹ *∃ G-inv't probability measure μ on X.* ($\mu(X) = 1$)

Corollary (⟺)

G amenable, acts on cpct metric space X (*by homeos*)
⟹ *∃ G-inv't probability measure μ on X.* ($\mu(X) = 1$)

Corollary

G amenable, acts on S^1 (*orient-preserving*) ⟹ *either:*
∃ finite orbit or abelianization of G is infinite.

Fact: G amenable, acts on $\mathbb{R}$, finitely generated ⟹ abelianization is ∞.

Corollary (⟺)

- *G amenable,*
- *acts by* (*cont*) *linear maps on vector space* $\left(\substack{\text{locally} \\ \text{convex}}\right)$,
- *C is a compact, convex, G-invariant subset* ($\neq \varnothing$)

⟹ *∃ fixed point in C.*

Corollary (Furstenberg)

$\Gamma \doteq \mathrm{SL}(3,\mathbb{Z}) \subset \mathrm{SL}(3,\mathbb{R}) = G$, $P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$ *(amenable)*.

Γ *acts on* S^1 $\implies$ $\exists$ Γ*-equivariant*
$\psi\colon G/P \to \mathrm{Prob}(S^1)$. *{ probability measures on* S^1 *}*

Theorem (Ghys)

ψ *is constant* *(a.e.)* ψ *is measurable*
$\therefore$ $\exists$ Γ*-inv't point in* $\mathrm{Prob}(S^1)$
$\therefore$ $\exists$ *finite orbit* *(since* $\Gamma/[\Gamma,\Gamma]$ *is finite)*.

Proof of Corollary.

$\{\,\Gamma$-equivariant $\psi\colon G \to \mathrm{Prob}(S^1)\,\}$ is convex, cpct.
P acts by translation (on domain).
$\implies$ P has fixed pt, which factors through G/P. □

Another definition of amenability

Notation

G f.g. $\implies$ $\exists \phi\colon F_n \twoheadrightarrow G$.
Let $B_r = \{\,$words of length $\le r\,\}$ in F_n.
(Note: $\#B_r \approx (2n-1)^r$.)

Example

$G = F_n \implies \#(B_r \cap \ker\phi) = 1 < (\#B_r)^{\epsilon}$.

$G = \mathbb{Z}^n \implies \#(B_r \cap \ker\phi) \approx \dfrac{\#B_r}{(2r+1)^n} = (\#B_r)^{1-\epsilon}$.

Theorem (R. I. Grigorchuk, J. M. Cohen**)**

G *amenable* $\iff \#(B_r \cap \ker\phi) \ge (\#B_r)^{1-\epsilon}$.
I.e., amenable groups have maximal cogrowth.

Exercises

1) *Examples of amenable groups:*
   a) finite groups are amenable ($S = G = F_n$)
   b) $\mathbb{Z}$ is amenable ($S = \{1\}$, $F_n = \{1,2,3,\cdots,n\}$)
   c) amenable $\times$ amenable is amenable
   d) abelian groups are amenable
   e) $N \triangleleft G$ with $N, G/N$ amen $\implies$ G amen
   f) solvable groups are amenable (!!!)
   g) subgrps, quotients of amen grps are amen
   h) grps of subexp'l growth are amenable
2) Grps with a nonabel free subgrp are not amen.
   Remark: (difficult) There exist nonamenable groups that do not have nonabelian free subgrps. In fact, torsion groups can be nonamen.

Optional exercises

3) locally amenable $\implies$ amenable
4) $\exists$ Følner sets $\implies$
   a) $\nexists$ Ponzi scheme.
   b) every bdd func on G has an avg value.
5) amenable $\implies$ $\nexists$ *paradoxical decomposition.*
   (If $G = \left(\coprod_{i=1}^m A_i\right) \coprod \left(\coprod_{j=1}^n B_j\right)$ (disjoint unions)
   and $g_1,\ldots,g_m,h_1,\ldots,h_n \in G$,
   show either $G \ne \bigcup_{i=1}^m g_i A_i$ or $G \ne \bigcup_{j=1}^n h_j B_j$.
6) Find an *explicit* paradoxical decomp of a free grp.
7) G acts on S^1, $\exists$ G-inv't probability measure
   $\implies$ $\exists$ finite orbit or $G/[G,G]$ is infinite.
8) G_1 has Ponzi, G_1 quasi-isom to $G_2 \Rightarrow G_2$ has Ponzi.

Related reading

- D. Morris: *Introduction to Arithmetic Groups* (preprint). (Has chapter on amenable groups.) `http://people.uleth.ca/~dave.morris/books/IntroArithGroups.html`
- É. Ghys: Groups acting on the circle. *L'Enseignement Mathématique* 47 (2001) 329–407. `http://retro.seals.ch/cntmng;?type=pdf&rid=ensmat-001:2001:47::210`
- D. W. Morris: Can lattices in $\mathrm{SL}(n,\mathbb{R})$ act on the circle?, in *Geometry, Rigidity, and Group Actions*, University of Chicago Press, Chicago, 2011. `http://arxiv.org/abs/0811.0051`

Further reading

Amenability:

- A. L. T. Paterson: *Amenability.* American Mathematical Society, Providence, RI, 1988.
- J.-P. Pier: *Amenable Locally Compact Groups.* Wiley, New York, 1984.
- S. Wagon: *The Banach-Tarski Paradox.* Cambridge U. Press, Cambridge, 1993.

Ponzi schemes:

- M. Gromov: *Metric structures for Riemannian and non-Riemannian spaces.* Birkhäuser, Boston, 1999. (See Lemma 6.17 and Exercise $6.17\frac{1}{2}$ on p. 328.)

Ghys' proof:

- É. Ghys: Actions de réseaux sur le cercle. *Invent. Math.* 137 (1999) 199–231.

A different way to show ψ *is constant:*

- U. Bader, A. Furman, A. Shaker: Superrigidity, Weyl groups, and actions on the circle (preprint). `http://arxiv.org/abs/math/0605276`