Some arithmetic groups that do not act on the circle

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Lecture 2: Proof for $SL(2,\mathbb{Z}[\alpha])$ using bounded generation

Recall
Γ = large arithmetic group
$\doteq SL(3,\mathbb{Z}),\ SL(2,\mathbb{Z}[\alpha]),$ etc.
α = irrational, algebraic, real

Conjecture
Γ *does not act on* $\mathbb{R}$. *(faithfully — no kernel)*
$\nexists$ *faithful homomorphism* $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$

Proposition (Witte [1994])
Γ *does not act on* $\mathbb{R}$ *if* $\Gamma \doteq SL(3,\mathbb{Z})$.

Theorem (Lifschitz-Morris [2004])
Γ *does not act on* $\mathbb{R}$ *if* $\Gamma \doteq SL(2,\mathbb{Z}[\alpha])$.

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Proof combines bdd generation and bdd orbits.
Unipotent subgroups: $\overline{U} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$, $\underline{V} = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}$.

Theorem (Carter-Keller-Paige, Lifschitz-Morris)
- $\overline{U}$ *and* $\underline{V}$ *boundedly generate* Γ *(up to finite index)*.
- Γ *acts on* $\mathbb{R} \implies \overline{U}$*-orbits (and* $\underline{V}$*-orbits) are bdd.*

Bounded generation by unip subgrps

Note: Invertible matrix $\rightsquigarrow$ Id by row operations.

Key fact: $g \in SL(2,\mathbb{Z}) \rightsquigarrow$ Id by integer ($\mathbb{Z}$) row ops.

Example

$$\begin{bmatrix} 13 & 31 \\ 5 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\therefore\ \overline{U}$ and $\underline{V}$ *generate* $SL(2,\mathbb{Z})$.

But # steps is *not bounded:*
$\overline{U}$ and $\underline{V}$ do **not** boundedly generate $SL(2,\mathbb{Z})$.

Key fact: $g \in SL(2,\mathbb{Z}) \rightsquigarrow$ Id by integer ($\mathbb{Z}$) row ops,
but # steps is *not bounded.*

Remark: In $SL(3,\mathbb{Z})$, # steps is bounded [Carter-Keller, 1983].

Theorem (Liehl [1984], Carter-Keller-Paige [1995?]**)**
For $\mathbb{Z}[\alpha]$ *row operations, # steps is bounded.*
$\exists n,\ \forall g \in SL(2,\mathbb{Z}[\alpha]),\ g = u_1 v_1 u_2 v_2 \cdots u_n v_n.$
I.e., $\overline{U}$ *and* $\underline{V}$ *boundedly gen* $\Gamma = SL(2,\mathbb{Z}[\alpha])$.
So $SL(2,\mathbb{Z}[\alpha]) = \overline{U}\,\underline{V}\,\overline{U}\,\underline{V} \cdots \overline{U}\,\underline{V}$.

Theorem (Liehl [1984])
$SL(2,\mathbb{Z}[1/p])$ *bddly gen'd by elem mats.*
I.e., $T \rightsquigarrow$ *Id by* $\mathbb{Z}[1/p]$ *col ops, # steps is bdd.*

Easy proof
Assume Artin's Conjecture:
$\forall r \neq \pm 1$, perfect square,
$\exists\ \infty$ primes q, s.t. r is primitive root modulo q:
$\{r, r^2, r^3, \ldots\} \bmod q = \{1,2,3,\ldots,q-1\}$
Assume $\exists q$ in every arith progression $\{a + kb\}$.

$\exists\ q = a + kb,\ p$ is a primitive root modulo q.

Theorem (Liehl [1984]**)**
$SL(2,\mathbb{Z}[1/p])$ *bddly gen'd by elem mats.*
I.e., $T \rightsquigarrow$ *Id by* $\mathbb{Z}[1/p]$ *col ops, # steps is bdd.*

Proof.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $q = a + kb$ prime, p is prim root

$\rightsquigarrow \begin{bmatrix} q & b \\ * & * \end{bmatrix}$ $p^\ell \equiv b \pmod q$; $p^\ell = b + k'q$

$\rightsquigarrow \begin{bmatrix} q & p^\ell \\ * & * \end{bmatrix}$ p^ℓ unit: can add *anything* to q

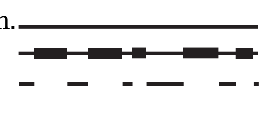
$\rightsquigarrow \begin{bmatrix} 1 & p^\ell \\ * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\square$

- Bdd generation: $\Gamma = \overline{U}\,\underline{V}\,\overline{U}\,\underline{V} \cdots \overline{U}\,\underline{V}$.
- Bdd orbits: $\overline{U}$-orbits and $\underline{V}$-orbits are bounded.

Corollary
$\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$ $\Rightarrow$ *every* Γ*-orbit on* $\mathbb{R}$ *is bdd*
$\Rightarrow \Gamma$ *has a fixed point.*

Corollary
Γ *cannot act on* $\mathbb{R}$.

Proof. Spse $\exists$ nontrivial action.
It has fixed points:
Remove them:
Take a connected component:
Γ acts on open interval ($\approx \mathbb{R}$) with no fixed pt. $\rightarrow\leftarrow$



Bounded orbits

Theorem (Lifschitz-Morris [2004])
$\Gamma = SL(2,\mathbb{Z}[1/p])$ *acts on* $\mathbb{R}$ $\Rightarrow$ *every* $\overline{U}$*-orbit bdd.*

$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}$, $\not{p} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$

Assume $\overline{U}$-orbit and $\underline{V}$-orbit of x not bdd above.
Assume $\not{p}$ fixes x. ($\not{p}$ does have fixed pts, so not an issue.)
- Wolog $\overline{u}(x) < \underline{v}(x)$.
- Then $\not{p}^n(\overline{u}(x)) < \not{p}^n(\underline{v}(x))$.
- LHS $= \not{p}^n(\overline{u}(x)) = (\not{p}^n \overline{u} \not{p}^{-n})(x) \to \overline{\infty}(x) \to \infty$.
- RHS $= \not{p}^n(\underline{v}(x)) = (\not{p}^n \underline{v} \not{p}^{-n})(x) \to \underline{0}(x) < \infty$.

$\rightarrow\leftarrow$ $\square$

Other arithmetic groups of higher rank

Proposition

Suppose $\Gamma_1 \subset \Gamma_2$.

- *If* Γ_2 *acts on* $\mathbb{R}$, *then* Γ_1 *acts on* $\mathbb{R}$.
- *If* Γ_1 *does not act on* $\mathbb{R}$, *then* Γ_2 *does not act on* $\mathbb{R}$.

Our methods require Γ to have a unipotent subgrp.
Such arithmetic groups are called *noncocompact*.

Theorem (Chernousov-Lifschitz-Morris [2008]**)**

Spse Γ *is a noncocompact arith group* *of higher rank*.
Then $\Gamma \mathrel{\dot{\supset}} \mathrm{SL}(2, \mathbb{Z}[\alpha])$
or noncocpct arith grp in $\mathrm{SL}(3,\mathbb{R})$ *or* $\mathrm{SL}(3,\mathbb{C})$.

Open Problem

Show noncocpct arith grps in $\mathrm{SL}(3,\mathbb{R})$ *and* $\mathrm{SL}(3,\mathbb{C})$ *cannot act on* $\mathbb{R}$.

Conjecture (Rapinchuk [~1990]**)**

These arith grps are boundedly generated by unips.

Rapinchuk Conjecture implies no action on $\mathbb{R}$
if Γ noncocompact of higher rank.

Cocompact case will require new ideas.

Open Problem

Find cocompact arithmetic group Γ, *such that finite-index subgroups of* Γ *do not act on* $\mathbb{R}$.

Exercises

1) Assume Γ boundedly generated (by cyclic subgrps).
   (I.e., $\Gamma = H_1 H_2 \cdots H_n$ with H_i cyclic.)
   If Γ acts by *isometries* on metric space X,
   and every cyclic subgroup has a bdd orbit on X,
   then every Γ-orbit on X is bounded.
2) $\mathrm{SL}(n,\mathbb{Z})$ bdd gen by unips
   $\implies$ $\mathrm{SL}(n+1,\mathbb{Z})$ bdd gen by unips (if $n \ge 2$).
3) Γ bdd gen (by cyclic subgrps)
   $\Leftrightarrow$ finite-index subgroup $\dot{\Gamma}$ bdd gen.
4) (harder) Free group F_2 *not* bdd gen (by cyclic subgrps).
5) $\overline{U}$ and $\underline{V}$ do not bddly gen $\mathrm{SL}(2,\mathbb{Z})$. (Use prev exer.)

Optional exercises

6) (harder) Assume Γ bdd gen (by cyclic subgrps).
   Show $\langle g^n \mid g \in \Gamma \rangle$ has finite index in Γ ($\forall n \in \mathbb{Z}^+$).
7) Assume:
   - Γ_1 and Γ_2 are arith subgrps of G_1 and G_2, resp.
   - G_1 and G_2 are simple Lie grps of higher real rank.
   - Γ_1 is cocompact, but Γ_2 is *not* cocompact.

   Use the Margulis Superrigidity Theorem to show Γ_2 is *not* isomorphic to a subgroup of Γ_1.

Related reading

- D. W. Morris: Can lattices in $\mathrm{SL}(n,\mathbb{R})$ act on the circle?, in *Geometry, Rigidity, and Group Actions*, University of Chicago Press, Chicago, 2011. http://arxiv.org/abs/0811.0051
- L. Lifschitz and D. Witte: Isotropic nonarchimedean S-arithmetic groups are not left orderable, *C. R. Math. Acad. Sci. Paris* 339 (2004), no. 6, 417–420. http://arxiv.org/abs/math/0405536

Further reading

- D. W. Morris: Bounded generation (unpublished). http://people.uleth.ca/~dave.morris/banff-rigidity/morris-bddgen.pdf
- D. W. Morris: Bounded generation of $\mathrm{SL}(n, A)$ (after D. Carter, G. Keller and E. Paige), *New York J. Math.* 13 (2007) 383–421. http://nyjm.albany.edu/j/2007/13-17.html
- L. Lifschitz and D. W. Morris: Bounded generation and lattices that cannot act on the line, *Pure and Applied Mathematics Quarterly* 4 (2008), no. 1, part 2, 99–126. http://arxiv.org/abs/math/0604612

- V. Chernousov, L. Lifschitz, and D. W. Morris: Almost-minimal nonuniform lattices of higher rank, *Michigan Mathematical Journal* 56, no. 2, (2008), 453–478. http://arxiv.org/abs/0705.4330
- A. Ondrus: Minimal anisotropic groups of higher real rank, *Michigan Math. J.* 60 (2011), no. 2, 355–397.