Some arithmetic groups that do not act on the circle

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Abstract. The group $SL(3,\mathbb{Z})$ cannot act (nontrivially) on the circle (by homeomorphisms). We will see that many other arithmetic groups also cannot act on the circle. The discussion will involve several important topics in group theory, such as amenability, Kazhdan's property (T), ordered groups, bounded generation, and bounded cohomology.

Lecture 1: Introduction

In **Geometric Group Theory** (and elsewhere):
Study group Γ by looking at spaces it can act on.
$X = \mathbb{H}^n$, CAT(0) cube cplx, Euclidean bldg, etc.

Question

¿ ∃ (*faithful*) *action of* Γ *on* X ? (faithful: no kernel)

In **these lectures**:

- Γ = *arithmetic group* $\doteq SL(n,\mathbb{Z})$ or ...
- X = simplest possible space
  = connected manifold of dim'n 1
  = *circle* or *line*

¿ ∃ (*almost faithful*) *homo* $\phi\colon \Gamma \to \mathrm{Homeo}_+(\mathbb{R})$? *or* $\mathrm{Homeo}_+(S^1)$?

Question

¿ ∃ (*faithful*) *action of* Γ *on* $\mathbb{R}$ *or* S^1? (Γ = *arith grp* $\doteq SL(n,\mathbb{Z})$)

Fact

$\dot\Gamma$ *acts on* $\mathbb{R}$ $\iff$ $\ddot\Gamma$ *acts on* S^1 (if $\Gamma \not\doteq SL(2,\mathbb{R})$)

Proof (⇒).

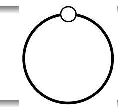
Γ acts on one-pt compactification of $\mathbb{R}$ $\approx S^1$. □

Theorem (Ghys, Burger-Monod, Bader-Furman)

Γ *acts on* S^1 $\implies$ ∃ *finite orbit* (if $\Gamma \not\doteq SL(2,\mathbb{R})$)
$\implies$ $\dot\Gamma$ *has a fixed point.*

Proof of Fact (⇐).

Γ acts on S^1 − (fixed pt) $\approx \mathbb{R}$. □



Question

¿ ∃ (*faithful*) *action of* Γ *on* $\mathbb{R}$? (Γ = *arith grp*)

Example

$SL(2,\mathbb{Z})$ does *not* act on $\mathbb{R}$.

Proof.

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 = I.$$ So $SL(2,\mathbb{Z})$ has elt's of finite order.

But $\mathrm{Homeo}_+(\mathbb{R})$ has no elt's of finite order:

$$\varphi(0) > 0 \implies \varphi^2(0) > \varphi(0) > 0 \implies \varphi^3(0) > 0 \implies \dots \implies \varphi^n(0) > 0.$$ □

Example

$\Gamma \doteq SL(2,\mathbb{Z})$ finite-index subgrp can be a *free group.*
Has *many* actions on $\mathbb{R}$.

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Example (Agol, Boyer-Rolfsen-Wiest)

$\Gamma \subset SO(1,3) \implies \dot\Gamma$ acts on $\mathbb{R}$ (because $\dot\Gamma \twoheadrightarrow \mathbb{Z}$)

Arith grps known to act on $\mathbb{R}$ are "small." ($\subset SO(1,n)$?)

Conjecture

Large arithmetic groups $\begin{pmatrix}\text{irreducible,}\\ \mathbb{R}\text{-rank} > 1\end{pmatrix}$ *cannot act on* $\mathbb{R}$

$\Gamma \doteq SL(3,\mathbb{Z})$ *or* $\Gamma \doteq SL(2,\mathbb{Z}[\alpha])$ *or* ...
α = *real, irrat alg'ic integer.*
$\Gamma \not\subset SO(1,n),\ SU(1,n),\ Sp(1,n),\ F_{4,1}$.

Algebraic translation of conjecture

Definition

Assume Γ acts (faithfully) on $\mathbb{R}$.
$a \prec b \iff a(0) < b(0)$ or ... (break ties)

Exercise

$\prec$ *is a total order on* Γ *that is left-invariant.*
$(a \prec b \implies ca \prec cb)$ *Hint: orient-pres:* $s\,x < y \implies c(x) < c(y)$.

Note: $a, b \succ e \implies ab \succ a \succ e$ and $e \succ a^{-1}$.

Exercise (assume Γ countable)

Γ *acts faithfully on* $\mathbb{R}$ $\iff$ ∃ *left-inv't order on* Γ.
Hint: $(\Gamma, \prec) \cong (\mathbb{Q}, <) \implies$ *Dedekind completion of* Γ *is* $\mathbb{R}$.

Conjecture

∄ *left-inv't order for* Γ = *large arith grp* $\doteq SL(3,\mathbb{Z})$, *etc.*

Proposition (Witte, 1994)

∄ *left-inv't order on* $\Gamma \doteq SL(3,\mathbb{Z})$.

Notation

$$H = \text{Heisenberg grp} = \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{bmatrix}.$$

$$x = \begin{bmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

Exercise

- $z = [x,y] = x^{-1}y^{-1}xy \in Z(H)$.
- (optional) *H has left-inv't order.*
  (N, G/N *left ord'ble* ⇒ G *left ord'ble* ["lexicographic order"])

Lemma

∀ *left-ordering of* $H = \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix}$
$\exists s \in \{x^{\pm 1}, y^{\pm 1}\}$, $z \ll s$, *i.e.,* $z^n \prec s$, $\forall n \in \mathbb{Z}$.

Proof.

Wolog $x, y, z \succ e$. (Replace x, y, z with inverse. Interchange x and y: $[y,x] = z^{-1}$.)

$z = x^{-1}y^{-1}xy \quad \Rightarrow xy = yxz$
$\Rightarrow x^n y^n = y^n x^n z^{n^2}$ (Recall $z \in Z(H)$)
$\Rightarrow x^n y^n x^{-n} y^{-n} = z^{-n^2}$. (quadratic)

Suppose $z^p \succ x$ and $z^q \succ y$.
Therefore $e \prec x^{-1}z^p,\ y^{-1}z^q,\ x,\ y$
$\Rightarrow e \prec x^n\,y^n\,(x^{-1}z^p)^n\,(y^{-1}z^q)^n$
$= x^n\,y^n\,x^{-n}\,y^{-n}\,z^{pn+qn}$
$= z^{-n^2} z^{(p+q)n} = z^{\text{negative}}$. →← □

Spse ∃ left-inv't order on $SL(3,\mathbb{Z}) = \begin{bmatrix} * & ① & ② \\ ④ & * & ③ \\ ⑤ & ⑥ & * \end{bmatrix}$.

⟨①, ②, ③⟩ = Heisenberg group.

There are actually 6 Heisenberg groups in Γ:
①, ②, ③, ②, ③, ④, ③, ④, ⑤
④, ⑤, ⑥, ⑤, ⑥, ①, ⑥, ①, ②.

①, ②, ③ = Heis grp ⇒ ② ≪ ① or ② ≪ ③.
Wolog ② ≪ ③.

②, ③, ④ = Heis grp ⇒ ③ ≪ ② or ③ ≪ ④.
Must have ③ ≪ ④. etc.

② ≪ ③ ≪ ④ ≪ ⑤ ≪ ⑥ ≪ ① ≪ ② ⇒ ② ≪ ②.

Conjecture

Γ *does not act on* $\mathbb{R}$ *if* Γ *= large arithmetic group.*

Proposition (Witte, 1994)

Γ *does not act on* $\mathbb{R}$ *if* $\Gamma \doteq \mathrm{SL}(3,\mathbb{Z})$ *or* $\mathrm{Sp}(4,\mathbb{Z})$ *or contains either.* *I.e.,* $\mathrm{rank}_{\mathbb{Q}}(\Gamma) \geq 2$.

Remark

- Proposition does not apply to $\mathrm{SL}(2,\mathbb{Z}[\alpha])$.
- G/Γ *compact* $\implies$ proposition *never* applies.

Open problem

Find arith group Γ, such that G/Γ is compact, and *finite-index subgroups* of Γ does not act on $\mathbb{R}$.

Conjecture

Γ *does not act on* $\mathbb{R}$ *(or* S^1*) if* Γ *= large arith group.*

Remark

Large arithmetic groups usually have *Kazhdan's Property* (T).

Open problem

¿ Groups with Kazhdan's Property (T) have no actions on $\mathbb{R}$ or S^1 ?

Theorem (Navas)

Groups with Kazhdan's Property (T) *have no* C^2 *actions on* S^1.

Conjecture

Γ *does not act on* $\mathbb{R}$ *(or* S^1*) if* Γ *= large arith group.*

Coming up:

- Tues: proof for $\mathrm{SL}(2,\mathbb{Z}[\alpha])$ (and others)
  - *bounded generation*
- Thurs: What is an *amenable* group?
  - used in proof of Ghys ($\exists$ finite orbit)
- Fri: Intro to *bounded cohomology* (quasimorphisms)
  - used in proof of Burger-Monod ($\exists$ finite orbit)

All lectures are essentially independent.

Exercises

1) Show Γ acts (faithfully) on $\mathbb{R}$ iff Γ is left-orderable.
   (For $\Rightarrow$, need to show that ties can be broken in a consistent way.)
2) In the Heisenberg group H, show:
   a) $z = [x,y] \in Z(H)$.
   b) $x^k y^\ell = y^\ell x^k z^{k\ell}$ for $k,\ell \in \mathbb{Z}$.
   c) H is left-orderable.
3) The proof that $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ is not left-orderable:
   a) Verify: $\langle ①, ②, ③ \rangle$, $\langle ②, ③, ④ \rangle$, etc are all isomorphic to the Heisenberg grp H.
   b) Generalize proof to finite-index subgroups.
4) Every fin gen free group has a faithful action on $\mathbb{R}$.

Optional exercises

5) Torsion-free, *abelian* groups are left-orderable.
6) Torsion-free, *nilpotent* groups are left-ord'ble.
7) (harder) Some torsion-free, *solvable* groups are *not* left-orderable!
8) *Locally* left-orderable $\implies$ left-orderable.
   (Assumption: every *finitely generated* subgrp of Γ is left-ord'ble.)
9) *Residually* left-ord'ble $\implies$ left-ord'ble.
   (Assumption: $\forall g \in \Gamma$, $\exists$ homo $\varphi\colon \Gamma \to H$, such that $\varphi(g) \neq e$ and H is left-orderable.)
10) *Locally indicable* $\implies$ left-orderable.
    (Assumption: the abelianization of every nontrivial, finitely generated subgroup is infinite.)
11) $\mathrm{SL}(3,\mathbb{Z})$ *not* isomorphic to subgrp of $\mathrm{SL}(2,\mathbb{Z}[\alpha])$

Related reading

- D. W. Morris: Can lattices in $\mathrm{SL}(n,\mathbb{R})$ act on the circle?, in *Geometry, Rigidity, and Group Actions*, University of Chicago Press, Chicago, 2011. http://arxiv.org/abs/0811.0051
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- S. Boyer, D. Rolfsen, and B. Wiest: Orderable 3-manifold groups, *Ann. Inst. Fourier (Grenoble)* 55 (2005), no. 1, 243–288. http://dx.doi.org/10.5802/aif.2098

Further reading

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- É. Ghys: Groups acting on the circle. *L'Enseignement Mathématique* 47 (2001) 329–407. http://retro.seals.ch/cntmng;?type=pdf&rid=ensmat-001:2001:47::210
- A. Navas: *Groups of Circle Diffeomorphisms,* Univ. of Chicago Press, 2011. http://arxiv.org/abs/math/0607481
- D. Morris: *Introduction to Arithmetic Groups* (preprint). http://people.uleth.ca/~dave.morris/books/IntroArithGroups.html