17.5 Triple Integrals

These are just like double integrals, but with another integration to perform. Although this is conceptually a simple extension of the idea, in practice it can get very complicated. For example, in two variables, there are just two different ways to integrate by iteration, depending on how we order the variables. But since there are six different orderings of three variables, there are now six different types of iterated integrals. In this section we shall illustrate the basic ideas, and not go into very much detail or complexity.

We begin with the definition of the integral of a function $w = f(x, y, z)$ over a region R in three dimensions. Cover R with a grid formed of the coordinate planes. and form the sum

$$\sum f(x, y, z)\Delta V$$

over all the cubes of the grid in R, where f is evaluated at a point of the cube, and ΔV is the volume of the cube. Taking the limit as the grid becomes infinitely fine, we obtain

$$\iiint_R f(x, y, z)dV \ . \tag{12}$$

As remarked above, triple integrals can be evaluated as iterated integrals.

Proposition 17.5. Suppose that $w = f(x, y, z)$ is a continuous function on the rectangular parallelipiped $R: \ a \le x \le b, \ c \le y \le d, \ p \le z \le q$. Then the triple integral (12) can be evaluated by iteration in any of six ways, depending upon which variable is chosen first. For example, if the variables are chosen in the order x, y, z, we have

$$\iiint_R f(x, y, z)dV = \int_a^b \Big[\int_c^d \Big[\int_p^q f(x, y, z)dz\Big]dy\Big]dx \ . \tag{13}$$

Example 20. Find the integral of $x^2 + yz$ over the region $R: \ 0 \le x \le 2, \ 1 \le y \le 4, 0 \le z \le 5$. If we iterate in the order x, y, z we get

$$\iiint_R (x^2 + yz)dV = \int_0^2 \Big[\int_1^4 \Big[\int_0^5 (x^2 + yz)dz\Big]dy\Big]dx \ .$$

The first integration gives

$$\int_0^5 (x^2 + yz)dz = (x^2 z + \frac{yz^2}{2})\Big|_0^5 = 5x^2 + \frac{25}{2}y \ .$$

The next integral is

$$\int_1^4 (5x^2 + \frac{25}{2}y)dy = (5x^2 y + \frac{25}{4}y^2)\Big|_1^4 = 20x^2 + 100 - 5x^2 + \frac{25}{4} = 15x^2 + \frac{375}{4} \ .$$

Finally, the last integration gives the desired result:

$$\int_0^2 (15x^2 + \frac{375}{4})dx = \frac{15x^3}{3} + \frac{375}{4}x\Big|_0^2 = \frac{455}{2} \ .$$

If we iterate in another order, the calculations are different (and could be easier or more difficult), but lead to the same answer. For example, let us iterate in the order y, z, x:

$$\iiint_R (x^2 + yz)dV = \int_1^4 \Big[\int_0^5 \Big[\int_0^2 (x^2 + yz)dx\Big]dz\Big]dy \ .$$

The first integral is

$$\frac{x^3}{3} + yzx\Big|_0^2 = 2yz + \frac{8}{3} \ .$$

The next integration leads to

$$yz^2 + \frac{8}{3}z\Big|_0^5 = 25y + \frac{40}{3} \ ,$$

and the final integration is

$$\int_1^4 (25y + \frac{40}{3})dy = \frac{25y^2}{2} + \frac{40}{3}\Big|_1^4 = \frac{455}{2} \ .$$

Now, for the integral over a general region R, the situation can easily become quite complicated. Here is one of the possibilities;

Proposition 17.6 Suppose that $w = f(x, y, z)$ is a continuous function on the region R. If R can be described as (x, y, z) such that $a \le x \le b$, $u(x) \le y \le v(x)$, $\psi(x, y) \le z \le \phi(x, y)$, then

$$\int\int\int f(x, y, z)dV = \int_a^b \Big[\int_{u(x)}^{v(x)} \Big[\int_{\psi(x,y)}^{\phi(x,y)} f(x, y, z)dz\Big]dy\Big]dx \ . \tag{14}$$

Each ordering of the variables leads to a different possibility of calculating the triple integral by iteration; some of which may not work, and some of which may be easier than others.

Example 21 Find the volume of the region in the first octant bounded by the plane $3x + y + 2z = 12$.

First we draw a diagram of the figure.

———Figure 16———

We see that we can order the variable in any way we please. Describing the region by the inequalities $0 \le x \le 4,\ 0 \le y \le 12 - 3x,\ 0 \le z \le 12 - 3x - y$ we obtain the iterated integral

$$Volume = \int_0^4 [\int_0^{12-3x} [\int_0^{12-3x-y} dz]dy]dx \ .$$

The first integration gives $12 - 3x - y$, and the second is

$$\int_0^{12-3x} (12 - 3x - y)dy = (12 - 3x)y - \frac{y^2}{2}\Big|_0^{12-3x} = \frac{(12 - 3x)^2}{2} \ ,$$

and finally

$$Volume = \int_0^4 (\frac{(12 - 3x)^2}{2})dx = -\frac{(12 - 3x)^3}{18}\Big|_0^4 = 96 \ .$$

Example 22. Find the z-coordinate of the centroid of the region of example 21.

To do that, we find the moment of the region about the plane $z = 0$:

$$Mom_{\{z=0\}} = \iiint_R zdV \ .$$

In general, if we have a choice, it is best to leave the most involved integration to the last; thus we may want to take the variables in the order z, x, y. This gives us the iterated integral

$$Mom_{\{z=0\}} = \int_0^6 [\int_0^{12-2z} [\int_0^{(12-2z-y)/3} zdx]dy]dz \ .$$

The first integral leads to $z(12 - 2z - y)/3$, and the second is

$$\frac{1}{3}\int_0^{12-2z} z(12 - 2z - y)dy = \frac{z}{3}((12 - 2z)y - \frac{y^2}{2})\Big|_0^{12-2z} = \frac{z}{6}(12 - 2z)^2 \ .$$

After a little algebra, we obtain

$$Mom_{\{z=0\}} = \frac{2}{3}\int_0^6 (36z - 12z^2 + z^3)dz = 72 \ .$$

Thus

$$\bar{z} = \frac{Mom_{\{z=0\}}}{Mass} = \frac{72}{96} = .75 \ .$$

Example 23. Let R be the region bounded by the xy-plane, the plane $x = 1$ and under the surface $z = x^2 - y^2$. Find the centroid of R.

First we draw a diagram of the figure.

———Figure 16———

We see that we can describe the region as: $0 \le x \le 1,\ -x \le y \le x,\ 0 \le z \le x^2 - y^2$. Thus

$$Volume = \int_0^1 [\int_{-x}^x [\int_0^{x^2-y^2} dz]dy]dx = \int_0^1 [\int_{-x}^x (x^2-y^2)dy = \frac{4}{3}\int_0^1 x^3 dx = \frac{1}{3}\ ,$$

$$Mom_{\{x=0\}} = \int_0^1 [\int_{-x}^x [\int_0^{x^2-y^2} xdz]dy]dx = \frac{4}{3}\int_0^1 x^4 dx = \frac{4}{15}\ .$$

The moment about the plane $y = 0$ is zero because of symmetry. Finally,

$$Mom_{\{z=0\}} = \int_0^1 [\int_{-x}^x [\int_0^{x^2-y^2} zdz]dy]dx = \frac{1}{2}\int_0^1 \int_{-x}^x (x^2-y^2)^2 dydx\ .$$

Working out the square, we find the inner integral to be

$$\int_{-x}^x (x^4 - x^2y^2 + y^4)dy = (x^4 y - \frac{2}{3}x^2y^3 + \frac{y^5}{5})\Big|_{-x}^x = 2x^5(1 - \frac{2}{3} + \frac{1}{5}) = \frac{16}{15}x^5\ .$$

Thus

$$Mom_{\{z=0\}} = \frac{8}{15}\int_0^1 x^5 dx = \frac{4}{45}\ ,$$

and

$$\bar{x} = \frac{4/15}{1/3} = \frac{4}{5}\ , \quad \bar{y} = 0\ . \quad \bar{z} = \frac{4/45}{1/3} = \frac{4}{15}\ .$$

Integration in other coordinates.

First, we recall that the volume of the parallelipiped spanned by the three vectors $\mathbf{U},\ \mathbf{V}, \mathbf{W}$ is $|\mathbf{U} \times (\mathbf{V} \cdot \mathbf{W})| = |\det(\mathbf{U}, \mathbf{V}, \mathbf{W})|$, where the determinant is that of the matrix whose rows are the components of $\mathbf{U}, \mathbf{V}, \mathbf{W}$. Now, suppose that we want to evaluate the triple integral

$$\iiint_R f dxdydz \tag{15}$$

where R corresponds to a region S in u, v, w space under a change of variables

$$\mathbf{X} = \mathbf{X}(u,v,w) = x(u,v,w)\mathbf{I} + y(u,v,w)\mathbf{J} + z(u,v,w)\mathbf{K}\ . \tag{16}$$

By a change of variables, we mean that the coordinates (u, v, w) in S uniquely determine a point in R; that is, in principle, we could solve the equations (16) for u, v, w in terms of x, y, z. Now, we can perform the integration in u, v, w space by selecting a fine grid in S, covering R by the image of the grid, and forming the sum $\sum f(X)\Delta V$, where ΔV is the volume of the figure in R corresponding to a typical cube in the grid on S. This will be an approximation to (15), which gets better as the grid gets finer.

If the grid is sufficiently fine, we can, at each cube, approximate by the linear approximation to the change of variables:

$$d\mathbf{X} = \mathbf{X}_u du + \mathbf{X}_v dv + \mathbf{X}_w dw \ .$$

A cube in S of side lengths du, dv, dw corresponds to the parallelipiped in R spanned by the vectors $\mathbf{X}_u du,\ \mathbf{X}_v dv,\ \mathbf{X}_w dw$, and the volume of that parallelipiped is

$$dV = |\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_w)| dudvdw \ . \tag{17}$$

The factor in (17) is called the **Jacobian** of the variable change, and is calculated as the (absolute value of the) determinant of the matrix whose rows are $\mathbf{X}_u,\ \mathbf{X}_v,\ \mathbf{X}_w$. This matrix is denoted by

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix} \ .$$

Proposition 17.7. Suppose that we are given the change of variables (16) so that the region R in (x,y,z) space corresponds to the region S in (u,v,w) space. Then we can calculate triple integrals by integrating over S as follows:

$$\int\!\!\int\!\!\int_R f(\mathbf{X}) dxdydz = \int\!\!\int\!\!\int_S f(\mathbf{X}(u,v,w)) \Big| \frac{\partial(x,y,z)}{\partial(u,v,w)} \Big| dudvdw \ .$$

Example 24. Find the volume of the region R given by the inequalities

$$0 \le z \le 4 \ , \quad 0 \le y+z \le 3 \ , \quad 0 \le x+y+z \le 5 \ .$$

This region is a parallelipiped, so by the appropriate change of coordinates, can be made to correspond to a rectangular parallelipiped. That is, we make the change of variables

$$u = x+y+z \ , \quad v = y+z \ , \quad w = z$$

so that R corresponds to the region S given by the inequalities $0 \le u \le 5,\ 0 \le v \le 3,\ 0 \le w \le 4$. Thus

$$Volume = \int\!\!\int\!\!\int_R dxdydz = \int\!\!\int\!\!\int_S \Big| \frac{\partial(x,y,z)}{\partial(u,v,w)} \Big| dudvdw \ .$$

Now, to calculate the Jacobian, we solve for x, y, z in terms of u, v, w:

$$x = u - v \ , \quad y = v - w \ , \quad z = w \ ,$$

so that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \ .$$

Thus

$$Volume = \int_0^5 \int_0^3 \int_0^4 dudvdw = 60 \ .$$

When we make a linear change of variables the Jacobian is a constant (but not always 1, as above). By using the following fact about determinants, we need not actually solve for x, y, z in terms of u, v, w:

(18)
$$\frac{\partial(x,y,z)}{\partial(u,v,w)}\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \ .$$

Example 25. Let $u = 3x - y + z,\ v = x + y + 2z,\ w = x - z$. Find the volume of the parallelipiped given by the inequalities $-2 \le u \le 3,\ 0 \le v \le 4,\ 2 \le u \le 7$.

We calculate

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} = -7 \ .$$

Thus, by (18),

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{7} \ ,$$

so that

$$Volume = \int_{-2}^3 \int_0^4 \int_2^7 \frac{1}{7} dudvdw = \frac{100}{7} \ .$$

Usually, we change coordinates only when the statement of the problem strongly suggests a different set of coordinates. Thus, if the problem has symmetry about the z-axis, we may change to spherical coordinates, and if there is symmetry about the origin, an change to spherical coordinates is indicated. The calculation of theJacobians for cylindrical and spherical coordinates leads to these expressions:

Cylindrical coordinates: $dV = rdrd\theta dz$.

Spherical coordinates: $dV = \rho^2 \sin\phi d\rho d\theta d\phi$.

Example 26. Find the volume and the centroid of the region bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = 0,\ z = 2$.

Here, because of the symmetry about the z-axis, we are led to cylindrical coordinates. In these coordinates, the region is given by $0 \le \theta \le 2\pi,\ 0 \le z \le 2,\ 0 \le r \le \sqrt{1+z^2}$. Thus

$$Volume = \iiint_R dxdydz = \iiint_R rdrdzd\theta = \int_0^{2\pi} [\int_0^2 [\int_0^{\sqrt{1+z^2}} rdr]dz]d\theta \ .$$

The inner integral is

$$\int_0^{\sqrt{1+z^2}} rdr = \frac{1+z^2}{2} \ ,$$

and the next integral is

$$\int_0^2 \frac{1+z^2}{2} dz = \frac{z}{2} + \frac{z^3}{6}\Big|_0^2 = \frac{7}{3} \ ,$$

and the final integration gives the volume as $14\pi/3$. Now, because of the symmetry, $\bar{x} = 0, \ \bar{y} = 0$. To calculate $\bar{z}$, we need

$$Mom_{\{z=0\}} = \int\int\int_R zrdrdzd\theta = \int_0^{2\pi} [\int_0^2 [\int_0^{\sqrt{1+z^2}} zrdr]dz]d\theta \ .$$

This time the first integral gives us $z(1+2^2)/2$, and the next is

$$\int_0^2 \frac{z+z^3}{2} dz = \frac{z^2}{4} + \frac{z^4}{8}\Big|_0^2 = 3 \ ,$$

so that $Mom_{\{z=0\}} = 6\pi$. Thus

$$\bar{z} = \frac{6\pi}{14\pi/3} = 9/7 \ .$$

Example 27. The region R between the spheres of radius 4 and 5 is filled with a material whose density is given by $\delta(x,y,z) = 1 + x^2 + y^2$. Find the mass of this region.

In spherical coordinates, the region is given by the inequalities $0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi, \ 4 \le \rho \le 5$, and $\delta = 1 + \rho^2 \sin^2\phi$. Thus

$$Mass = \int_0^{2\pi} [\int_0^{\pi} [\int_4^5 (1+\rho^2\sin^2\phi)\rho^2 \sin\phi d\rho]d\phi]d\theta \ .$$

The innermost integral is

$$\int_4^5 (1+\rho^2\sin^2\phi)\rho^2\sin\phi d\rho = \frac{61}{3}\sin\phi + \frac{2101}{5}\sin^3\phi \ .$$

Now, to integrate with respect to ϕ, we calculate

$$\int_0^{\pi} \sin\phi d\phi = 2 \ , \quad \int_0^{\pi} \sin^3\phi d\phi = \int_0^{\pi} (1-\cos^2\phi)\sin\phi d\phi = \frac{4}{3} \ ,$$

and the integration with respect to θ introduces a factor of 2π. Thus

$$Mass = 2\pi(\frac{61}{3}(2) + \frac{2101}{5}\frac{4}{3}) = 1201.87\pi = 3775.67 \ .$$