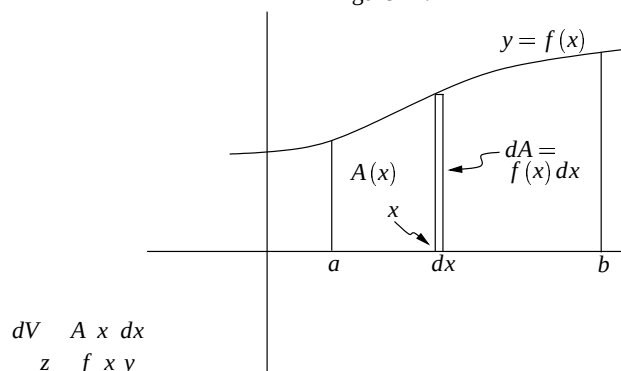


## Multiple Integration

### §17.1. Integration on Planar Regions

Integration of functions in several variables is done following the ideas of “accumulation” introduced in Chapter 4. There, for example, we calculated the area under a curve  $y = f(x)$  as  $x$  ranges from  $x = a$  to  $x = b$  by accumulating the area as we swept the region out along the  $x$ -axis from  $a$  to  $b$ . If we define the function  $A(x)$  to be the area swept out up to the value  $x$ , then we calculated, using figure 17.1,  $dA = f(x)dx$ : the increment in area is equal to the increment in  $x$  times the height of the rectangle at  $x$ .

Figure 17.1



We will show that the same idea, now for calculating volumes, works in two dimensions, leading to what is called an “iterated integral”. In section 17.3 we shall give a more formal definition of the double integral, and then see that its computation uses the technique of iteration introduced in this section.

**Definition 17.1** Let  $f(x,y)$  be a function defined on a region  $R$  in the plane.

a) If  $f(x,y)$  is positive for all  $(x,y)$  in the region  $R$ , then the volume of the solid lying over the region  $R$  and under the graph  $z = f(x,y)$  is the **double integral** of  $f$  over  $R$ , denoted

$$(17.1) \quad \iint_R f(x,y) dA .$$

b) For a general  $f$ , the double integral (17.1) is the signed volume bounded by the graph  $z = f(x, y)$  over the region; that is, the volume of the part of the solid below the  $xy$ -plane is taken to be negative.

**Proposition 17.1** (Iterated Integrals). We can compute  $\int \int_R f dA$  on a region  $R$  in the following way.

a) Suppose  $R$  lies between the lines  $x = a$  and  $x = b$ . For each  $x$  between  $a$  and  $b$ , let  $A(x)$  be the signed area of the region defined by the graph of  $z = f(x, y)$  over  $R$ , with  $x$  held constant (see figure 17.2). Then

$$(17.2) \quad \int \int_R f(x, y) dA = \int_a^b A(x) dx = \int_a^b \left[ \int f(x, y) dy \right] dx.$$

b) Suppose  $R$  lies between the lines  $y = c$  and  $x = d$ . For each  $y$  between  $c$  and  $d$ , let  $A(y)$  be the signed area of the region defined by the graph of  $z = f(x, y)$  over  $R$ , with  $y$  held constant (see figure 17.3). Then

$$(17.3) \quad \int \int_R f(x, y) dA = \int_c^d A(y) dy = \int_c^d \left[ \int f(x, y) dx \right] dy.$$

Figure 17.2

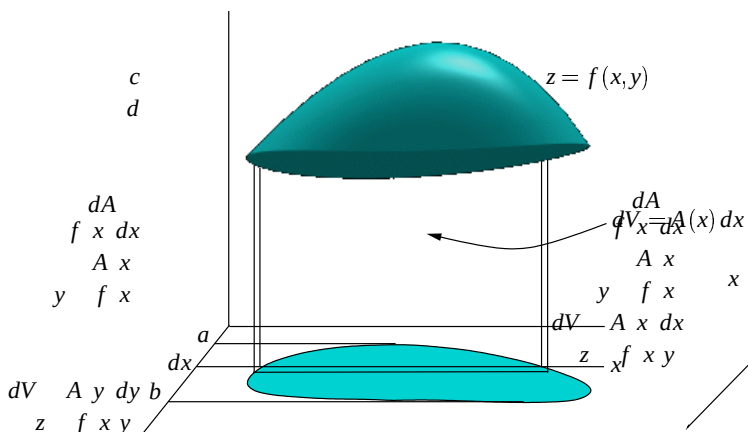
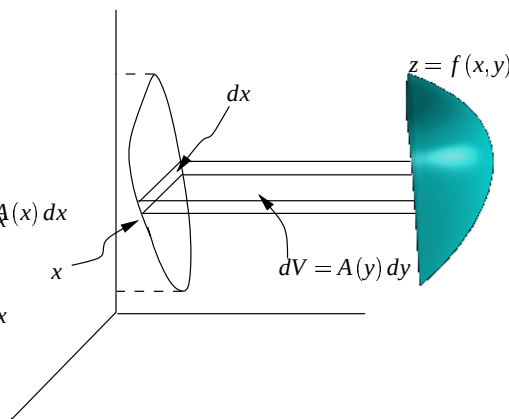


Figure 17.3



For (17.2), we sweep out the volume along the  $x$ -axis, letting  $V(x)$  be the volume accumulated from  $a$  to  $x$ . Now, an approximation to the increment  $\Delta V$  in  $V$  by moving a small distance  $\Delta x$  is the product of the cross-sectional area  $A(x)$  with  $\Delta x$  (see figure 17.2). This leads to the differential equation  $dV = A(x) dx$ , which is just the differential form of the first equality of (17.2). But, the area  $A(x)$  is just  $\int f(x, y) dy$ . (17.3) is demonstrated in the same way by sweeping out in the direction of the  $y$ -axis.

We have left out the limits of integration in the inner integrals of equations (17.2) and (17.3) for simplicity of notation. Determining them could be quite complicated. We start with some simple cases.

**Example 17.1** Find the volume of the solid over the rectangle  $0 \leq x \leq 1$ ,  $0 \leq y \leq 3$  and bounded by the  $xy$ -plane and the plane  $z = x + y$  (see figure 17.4).

For  $x$  between 0 and 1, we calculate the area of the section of the solid by the plane with  $x$  fixed;

$$(17.4) \quad A(x) = \int_0^3 z dy = \int_0^3 (x + y) dy = \left( xy + \frac{y^2}{2} \right) \Big|_0^3 = 3x + \frac{9}{2}.$$

Then the volume is

$$(17.5) \quad \int_0^1 A(x) dx = \int_0^1 (3x + \frac{9}{2}) dx = (\frac{3}{2}x^2 + \frac{9}{2}x) \Big|_0^1 = 6.$$

Figure 17.4

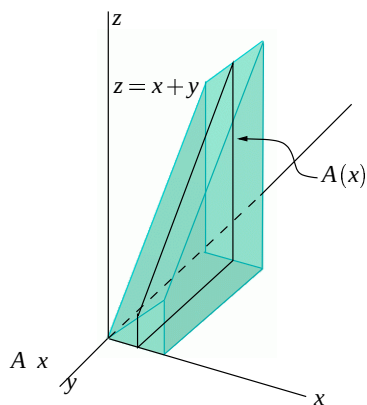
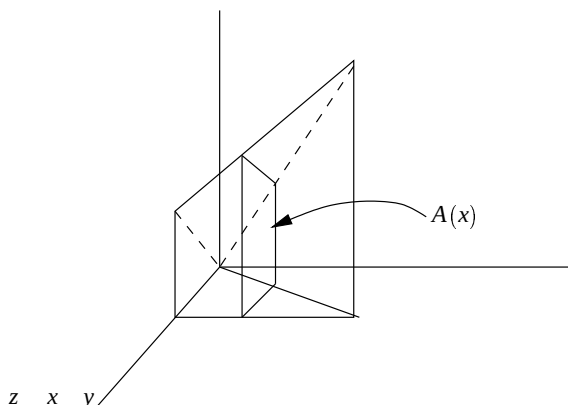


Figure 17.5



**Example 17.2** Suppose that a house is situated in one corner of a rectangular plot of land 300 feet by 200 feet. The contour of this plot of land is given by the equation  $E(x, y) = 10^{-4}(x^2 - xy/2)$  where the house is situated at the origin and the  $x$ -axis is the 300 foot length. It is desired to level this property at the level of the house. How much fill has to be removed (or brought in) to accomplish this?

We want to know the difference between the volume of land above house level and that below house level; that is, we want to find the signed volume determined by the graph of  $z = E(x, y)$  over the plot of land  $R$ . We have

$$(17.6) \quad \iint_R E(x, y) dA = \int_0^{300} A(x) dx,$$

where  $A(x)$  is the cross-sectional (signed) area of the terrain profile on a section perpendicular to the  $x$  axis at a distance  $x$  from the house. This is

$$(17.7) \quad \int_0^{200} E(x, y) dy = 10^{-4} \int_0^{200} (x^2 - \frac{xy}{2}) dy = 10^{-4} (x^2 y - \frac{xy^2}{4}) \Big|_0^{200} = .02x^2 - x.$$

Then

$$(17.8) \quad \iint_R E(x, y) dA = \int_0^{300} [\int_0^{200} E(x, y) dy] dx = \int_0^{300} (.02x^2 - x) dx = (.02 \frac{x^3}{3} - \frac{x^2}{2}) \Big|_0^{300}.$$

This evaluates to  $13.5 \times 10^4$  cubic feet which, because it is positive, have to be removed. Were we to ask for the average elevation of the property above house level, we divide this by the area of the property, getting  $13.5 \times 10^4 / (6 \times 10^4) = 2.25$  feet.

**Example 17.3** Find the volume under the plane  $z = x + 2y + 1$  over the triangle bounded by the lines  $y = 0$ ,  $x = 1$ ,  $y = 2x$  (see figure 17.5).

If we sweep out along the  $x$ -axis, we can calculate the volume as  $\int_0^1 A(x)dx$ , where, for fixed  $x$ ,  $A(x)$  is the area under the curve  $z = x + 2y + 1$  over the line segment at  $x$  in the triangle. This is the line from  $y = 0$  to  $y = 2x$ . Thus

$$(17.9) \quad A(x) = \int_0^{2x} (x + 2y + 1)dy = (xy + y^2 + y) \Big|_0^{2x} = 2x^2 + 4x^2 + 2x = 6x^2 + 2x.$$

Then the volume is

$$(17.10) \quad \text{Volume} = \int_0^1 \left[ \int_0^{2x} (x + 2y + 1)dy \right] dx = \int_0^1 (6x^2 + 2x)dx = (2x^3 + x^2) \Big|_0^1 = 3.$$

For confirmation that we can calculate integrals by iterating in either order, we'll calculate the volume by sweeping out along the  $y$  axis first. Now,  $y$  ranges from 0 to 2, and for fixed  $y$ ,  $x$  ranges from  $y/2$  to 1. This computation leads to

$$(17.11) \quad \text{Volume} = \int_0^2 \left[ \int_{y/2}^1 (x + 2y + 1)dx \right] dy.$$

The inner integral is

$$(17.12) \quad \int_{y/2}^1 (x + 2y + 1)dx = \left( \frac{x^2}{2} + 2xy + x \right) \Big|_{y/2}^1 = \frac{3}{2} + \frac{3}{2}y - \frac{9}{8}y^2.$$

We then get

$$(17.13) \quad \text{Volume} = \int_0^2 \left( \frac{3}{2} + \frac{3}{2}y - \frac{9}{8}y^2 \right) dy = \left( \frac{3}{2}y + \frac{3}{4}y^2 - \frac{3}{8}y^3 \right) \Big|_0^2 = 3 + \frac{3}{4} - \frac{3}{4} = 3.$$

This last example illustrates two important considerations:

1. We can try to integrate by sweeping out along either coordinate axis, and one order of integration may be simpler than the other.
2. To integrate a function  $f$  over a domain, first draw a diagram of the domain to determine the preferred (sometimes, only possible) order of integration. If we try sweeping out along the  $x$ -axis, first determine the range in the variable  $x$ , and then assure yourself that any line  $x = \text{constant}$  intersects the region in an interval. If not, sweep in the other direction, so that any line  $y = \text{constant}$  is an interval. Of course, both attempts may fail; we'll look into this in the next section. If one or the other criterion holds, we say the region is regular.

**Definition 17.2** A region in the plane is **regular** if it can be described in either of these two ways:

**(type 1):** as the set of  $(x, y)$  where  $x$  runs from  $a$  to  $b$ , and for each such  $x$ ,  $y$  lies between  $\phi(x)$  and  $\psi(x)$ ;

**(type 2):** as the set of  $(x, y)$  where  $y$  runs from  $c$  to  $d$ , and for each such  $y$ ,  $x$  lies between  $\mu(y)$  and  $\nu(y)$ .

Figure 17.6: Type 1

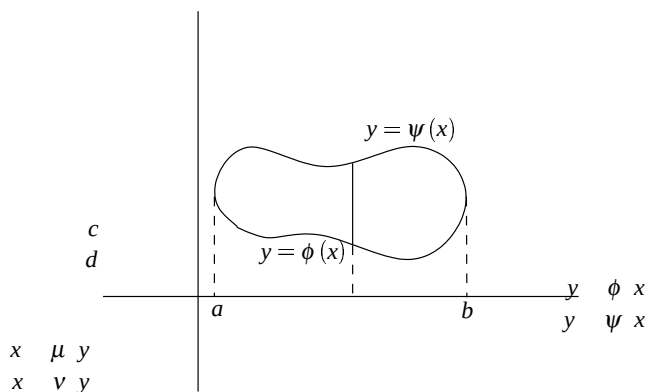
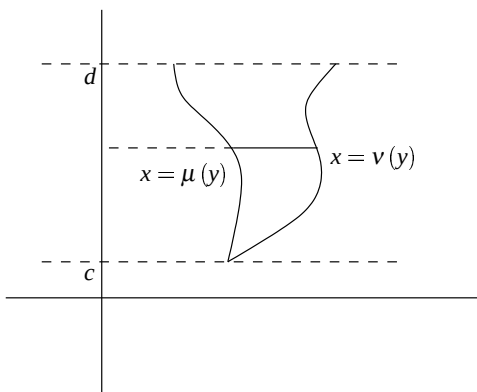


Figure 17.7: Type 2



**Proposition 17.2** Suppose that  $f$  is defined over a regular region  $R$ . Then we can calculate the double integral  $\iint_R f dA$  as an **iterated** integral:

$$(17.14) \quad \iint_R f dA = \int_a^b \left[ \int_{\phi(x)}^{\psi(x)} f(x, y) dy \right] dx \quad (\text{type 1}) ,$$

$$(17.15) \quad \iint_R f dA = \int_c^d \left[ \int_{\mu(y)}^{\nu(y)} f(x, y) dx \right] dy \quad (\text{type 2}) .$$

If  $R$  is regular of both types 1 and 2, then we can calculate either way, whichever is more convenient.

**Example 17.4** Let  $R$  be the region in the first quadrant between the curves  $y = 12x(1-x)$  and  $y = x(1-x)$ . Find  $\iint_R xy dA$ .

Draw the figure (see figure 17.8). This is a type 1 region, where for any  $x$ ,  $y$  lies on the interval from  $x(1-x)$  to  $12x(1-x)$ . This interval shrinks to a point when  $x = 0$  or  $1$ , so the range of  $x$  is  $0 \leq x \leq 1$ . Thus

$$(17.16) \quad \iint_R xy dA = \int_0^1 \left[ \int_{x(1-x)}^{12x(1-x)} xy dy \right] dx .$$

The inner integral is

$$(17.17) \quad x \frac{y^2}{2} \Big|_{x(1-x)}^{12x(1-x)} = \frac{x}{2} (144x^2(1-x)^2 - x^2(1-x)^2) = \frac{143}{2} (x^2 - 2x^3 + x^4) .$$

Thus

$$(17.18) \quad \iint_R xy dA = \frac{143}{2} \int_0^1 (x^2 - 2x^3 + x^4) dx = \frac{143}{2} \left( \frac{1}{3} - 2\frac{1}{4} + \frac{1}{5} \right) = \frac{143}{60} .$$

Figure 17.8

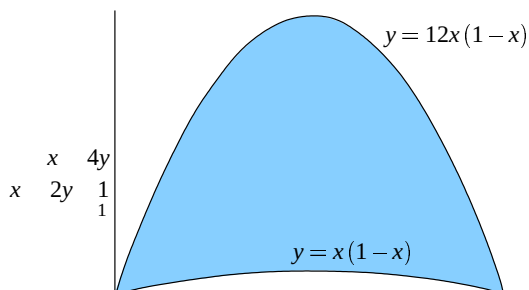
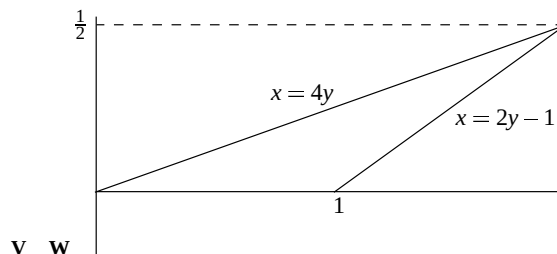


Figure 17.9



**Example 17.5** Let  $R$  be the region in the first quadrant between the lines  $x - 4y = 0$  and  $x - 2y = 1$ . Evaluate

$$(17.19) \quad \iint_R \frac{x}{1+y^2} dA.$$

Draw the figure (see figure 17.9). This is not a type 1 region, but it is type 2. The point of intersection of the lines is the simultaneous solution of the two equations”  $x = 2$ ,  $y = 1/2$ . Thus the region can be described as  $0 \leq y \leq 1/2$ , and for any such  $y$ ,  $4y \leq x \leq 1 + 2y$ . Thus

$$(17.20) \quad \iint_R \frac{x}{1+y^2} dA = \int_0^{1/2} \left[ \int_{4y}^{1+2y} \frac{x}{1+y^2} dx \right] dy.$$

The inner integral is

$$(17.21) \quad \frac{1}{1+y^2} \frac{x^2}{2} \Big|_{4y}^{1+2y} = \frac{(1+2y)^2 - 16y^2}{2(1+y^2)} = \frac{1+4y-12y^2}{2(1+y^2)}.$$

We now want to integrate this from 0 to  $1/2$ . To do so, we will need to do the long division:

$$(17.22) \quad \frac{1+4y-12y^2}{2(1+y^2)} = -6 + 2 \frac{x}{1+x^2} + \frac{13}{2(1+x^2)}.$$

Finally, the value of the double integral we seek is

$$(17.23) \quad \int_0^{1/2} \left( -6 + \frac{2x}{1+x^2} + \frac{13}{2(1+x^2)} \right) dx = \left[ -6x + \ln(1+x^2) + \frac{13}{2} \arctan x \right]_0^{1/2}$$

which evaluates to  $-3 + \ln(1.25) + 6.5 \arctan(.5) = .2368$ .

## §17.2. Applications

Just as in one dimension (refer to Chapter 5), we can apply integration to any concept which is “accumulative”: that is, its value over a whole is the sum of its value of the parts.

## §17.2.1 Average

Let  $R$  be a region in the plane. Its area is  $\int \int_R dA$ . If  $f$  is a function defined on  $R$ , its **average value** is

$$(17.24) \quad f_{ave} = \frac{\int \int_R f dA}{\int \int_R dA}.$$

We already calculated an average, in example 17.2. Here is another example.

**Example 17.6** We can model the state of Kansas as the rectangle  $K : 0 \leq x \leq 500, 0 \leq y \leq 300$ , where the units are miles. On a cold winter's day the temperature at  $(x, y)$  was  $T(x, y) = .01x(1 - 10^{-5}xy)$  degrees Celsius. What was the average temperature in the state?

To answer the question we consider  $T(x, y)$  as a measure of the amount of “heat” at  $(x, y)$ . Then the total heat in the state is

$$(17.25) \quad H = \int \int_K T(x, y) dA = \int_0^{500} \int_0^{300} (10^{-2}x - 10^{-7}x^2y) dy dx.$$

The inner integral is

$$(17.26) \quad \int_0^{300} (10^{-2}x - 10^{-7}x^2y) dy = (10^{-2}xy - 10^{-7}x^2 \frac{y^2}{2})_0^{300} = 3x - 4.5 \times 10^{-3}x^2.$$

Finally

$$(17.27) \quad H = \int_0^{500} (3x - 4.5 \times 10^{-3}x^2) dx = \left( \frac{3}{2}x^2 - 1.5 \times 10^{-3}x^3 \right)_0^{500} = 18.75 \times 10^4.$$

Since the area of Kansas is  $15 \times 10^4$  square miles, the average is

$$(17.28) \quad \frac{H}{A} = \frac{18.75 \times 10^4}{15 \times 10^4} = 1.25^\circ \text{C}.$$

## §17.2.2 Mass

A thin plate covering a region  $R$  in the plane is called a **lamina**. We suppose the lamina is filled with some inhomogeneous material, whose density over the point  $(x, y)$  is  $\delta(x, y)$ . Then the **total mass** of the plate is

$$(17.29) \quad \text{Mass} = \int \int_R \delta dA.$$

**Example 17.7** Suppose the rectangle  $R: 0 \leq x \leq 3, 0 \leq y \leq 4$  is filled with an inhomogeneous fluid whose density at the point  $(x, y)$  is  $\delta(x, y) = xy/6$ . Find the total mass.

$$(17.30) \quad \text{Mass} = \int \int_R \frac{xy}{6} dA = \frac{1}{6} \int_0^3 \left[ \int_0^4 xy dy \right] dx.$$

The inner integral is

$$(17.31) \quad \int_0^4 xy dy = x \frac{y^2}{2} \Big|_0^4 = 8x ,$$

so

$$(17.32) \quad \text{Mass} = \frac{1}{6} \int_0^3 8x dx = \frac{1}{6} (4x^2)_0^3 = 6 .$$

### §17.2.3 Moment

Given a mass  $m$  at a point  $P$ , its **moment** about an axis  $L$  is  $d \cdot m$ , where  $d$  is the distance of  $P$  to the axis  $L$ . It is Archimedes' principle that moments are additive, so if we have a lamina covering a region  $R$ , we can approximate its total moment about an axis  $L$  by covering it with a grid of small rectangles, and adding up the moments of each of the rectangles. In the limit this becomes a double integral: the **moment of the lamina  $R$  about the axis  $L$**  is

$$(17.33) \quad \text{Mom}_L = \iint_R d_L \delta dA ,$$

where  $\delta$  is the density function and  $d_L$  is the signed distance from the axis  $L$ . We take the signed distance because, when the axis passes through the region the moments on one side have the opposite effect of the moments on the other. The lamina is **balanced** along the axis  $L$  if  $\text{Mom}_L = 0$ . The **center of mass** of  $R$  is a point  $\bar{P} = (\bar{x}, \bar{y})$  such that  $R$  is balanced along every axis through  $P$ . To calculate these, use the following:

$$(17.34) \quad \text{Mom}_{x=0} = \iint_R x \delta dA$$

$$(17.35) \quad \text{Mom}_{y=0} = \iint_R y \delta dA$$

$$(17.36) \quad \bar{x} = \frac{\text{Mom}_{x=0}}{\text{Mass}} , \quad \bar{y} = \frac{\text{Mom}_{y=0}}{\text{Mass}} .$$

**Example 17.8** Find the center of mass of the rectangle in example 17.7.

$$(17.37) \quad \text{Mom}_{x=0} = \iint_R x \delta dA = \frac{1}{6} \int_0^3 \left[ \int_0^4 x^2 y dy \right] dx = \frac{1}{6} \int_0^3 8x^2 dx = 12 .$$

$$(17.38) \quad \text{Mom}_{y=0} = \iint_R y \delta dA = \frac{1}{6} \int_0^3 \left[ \int_0^4 xy^2 dy \right] dx = \frac{1}{6} \int_0^3 \frac{64x}{3} dx = 16 .$$

In example 17.7, we found the mass to be 6, so the center of mass is at  $(12/6, 16/6) = (2, 2.67)$ .



Finally (this concept is used when considering the rotation of a lamina  $R$  in motion), the **moment of inertia** about an axis  $L$  is  $\iint_R d_L^2 \delta dA$ , giving

$$(17.39) \quad I_{x=0} = \iint_R x^2 \delta dA ,$$

$$(17.40) \quad I_{y=0} = \iint_R y^2 \delta dA .$$

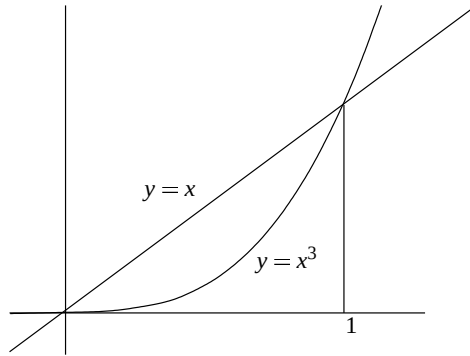
If we are interested in rotation of the region in its plane about the origin, we take as the squared term the distance to the origin (which can also be considered as rotation about the  $z$ -axis). Thus the moment of inertia of the region about the origin is

$$(17.41) \quad I_O = \iint_R (x^2 + y^2) \delta dA$$

**Example 17.9** Consider a lamina over the region  $R$  in the first quadrant bounded by the curves  $y = 2x$  and  $y = x^3$ . The density of the material in the lamina is  $\delta(x, y) = y$ . Find the center of mass of  $R$  and the moment of inertia about the origin.

First, we draw a diagram of the region  $R$  (figure 17.10).

Figure 17.10



$$(17.42) \quad \text{Mass} = \iint_R \delta dA = \int_0^1 \left[ \int_{x^3}^{2x} y dy \right] dx = \int_0^1 \frac{1}{2} (x^2 - x^6) dx = \frac{1}{2} \left[ \frac{1}{3} - \frac{1}{7} \right] = \frac{2}{21} .$$

$$(17.43) \quad \text{Mom}_{x=0} = \iint_R x \delta dA = \int_0^1 \left[ \int_{x^3}^{2x} x y dy \right] dx = \int_0^1 \frac{1}{2} (x^3 - x^7) dx = \frac{1}{2} \left[ \frac{1}{4} - \frac{1}{8} \right] = \frac{1}{16} .$$

$$(17.44) \quad \text{Mom}_{y=0} = \iint_R y \delta dA = \int_0^1 \left[ \int_{x^3}^{2x} y^2 dy \right] dx = \int_0^1 \frac{1}{3} (x^3 - x^9) dx = \frac{1}{3} \left[ \frac{1}{4} - \frac{1}{10} \right] = \frac{1}{20} .$$

Thus the center of mass is at  $(\bar{x}, \bar{y})$ , where

$$(17.45) \quad \bar{x} = \frac{1/16}{2/21} = \frac{21}{32} , \quad \bar{y} = \frac{1/20}{2/21} = \frac{21}{40} .$$

Finally, the moment of inertia about the origin is

$$(17.46) \quad I_O = \iint_R (x^2 + y^2) \delta dA = \int_0^1 \left[ \int_{x^3}^x (x^2 + y^2) y dy \right] dx .$$

The inner integral is

$$(17.47) \quad \int_{x^3}^x (x^2 y + y^3) dy = \left( x^2 \frac{y^2}{2} + \frac{y^4}{4} \right) \Big|_{x^3}^x = \frac{3x^4}{4} - \frac{x^8}{2} - \frac{x^{12}}{4} ,$$

so, finally

$$(17.48) \quad I_O = \int_0^1 \left( \frac{3x^4}{4} - \frac{x^8}{2} - \frac{x^{12}}{4} \right) dx = \frac{44}{585} .$$

### §17.3. Theoretical Considerations

Let  $f(x, y)$  be a function defined for all  $(x, y)$  in a region  $R$  in the plane. The integral of  $f$  over  $R$  is defined, as in one variable, as a limit of approximating sums. Recall that in one variable, the interval of integration was partitioned into small interval, and for each interval we formed the product  $f(\bar{x})\Delta x$ , where  $\bar{x}$  is a point in the interval, and  $\Delta x$  is the length of the interval. The sum  $\sum f(\bar{x})\Delta x$  over all intervals is the approximation to the integral for this partition. Now we do the same thing, taking small rectangles instead of intervals.

Select a grid  $G$  of horizontal and vertical lines, and let  $|G|$  represent the maximal distance between any two successive lines. Given such a grid, let  $G(R)$  be the set of rectangles wholly contained inside the region  $R$ . Then form the sum

$$(17.49) \quad \sum f(\bar{x}, \bar{y}) \Delta A$$

over all rectangles in  $G(R)$ . Here,  $\Delta A$  is the area of one such rectangle, and  $(\bar{x}, \bar{y})$  is any point in the rectangle. This sum is called the Riemann sum for  $f$  on  $R$  over the grid  $G$ . If the grid is very fine (that is,  $|G|$  is very small), this is an approximation to the integral.

**Definition 17.3**  $f$  is said to be **integrable** over the region  $R$  if

$$(17.50) \quad \lim_{|G| \rightarrow 0} \sum f(x, y) \Delta A$$

exists. In this case, the limit is the **definite integral** of  $f$  over  $R$ , denoted

$$(17.51) \quad \iint_R f(x, y) dA .$$

The question of integrability of a function over a region  $R$  is very difficult, but for reasonable functions and regions is covered by this

**Theorem 17.1** Suppose that  $R$  is a rectangle ( $a \leq x \leq b$ ,  $c \leq y \leq d$ ), and  $f$  is bounded and continuous on  $R$  except along a finite set of smooth curves, then  $f$  is integrable on  $R$ ; that is, the limit (17.50) exists.

To evaluate the integral, we reduce the problem to the one variable calculus, by adding the terms in (17.50) first in the vertical columns, and then adding the sums corresponding to the columns. Suppose then, that  $R$  is the rectangle  $a \leq x \leq b$ ,  $c \leq y \leq d$ . For each column between two grid lines on the  $x$ -axis, form the sum  $\sum f(x,y)\Delta y$ . Now take these expressions, multiply by the width of the column ( $\Delta x$ ) and sum over all columns, obtaining

$$(17.52) \quad \sum \left[ \sum f(x,y)\Delta y \right] \Delta x .$$

Taking the limits (as  $|G| \rightarrow 0$ , which is the same as  $\Delta x \rightarrow 0, \Delta y \rightarrow 0$ ) we obtain

$$(17.53) \quad \iint_R f(x,y) dA = \int_a^b \left[ \int_c^d f(x,y) dy \right] dx ,$$

where the inner integral is taken with  $x$  treated as a constant. Of course, we can sum first over the rows, and then add these sums up the  $y$ -axis:

$$(17.54) \quad \iint_R f(x,y) dA = \int_c^d \left[ \int_a^b f(x,y) dx \right] dy .$$

Now, what do we do if the region  $R$  is more general than a rectangle? There is no general procedure; but for regular domains, we can deduce proposition 17.2 from the above theorem. Suppose, for example that  $R$  is a type 1 domain, bounded on the left and right by lines  $x = a$ ,  $x = b$ , and below by a curve  $y = \phi(x)$ , and above by a curve  $y = \psi(x)$ . Then we can enclose the region  $R$  in a rectangle  $R_0$  bounded by the lines  $x = a$ ,  $x = b$ ,  $y = c$ ,  $y = d$ . If we extend  $f$  to all of  $R_0$  by defining it to be zero outside  $R$ , this extension is continuous on  $R_0$  except along the curve  $y = \phi(x)$ ,  $y = \psi(x)$ , so the theorem applies. But now

$$(17.55) \quad \int_c^d f(x,y) dy = \int_{\phi(x)}^{\psi(x)} f(x,y) dy$$

since  $f(x,y) = 0$  for  $y < \phi(x)$  or  $y > \psi(x)$ . Thus we have

$$(17.56) \quad \iint_R f(x,y) dA = \int_a^b \left[ \int_{\phi(x)}^{\psi(x)} f(x,y) dy \right] dx .$$

Similarly, for a type 2 region  $R$ , bounded below and above by lines  $y = c$ ,  $y = d$ , and on the left by a curve  $x = \mu(y)$ , and on the right by a curve  $x = \nu(y)$ , then the result is

$$(17.57) \quad \iint_R f(x,y) dA = \int_c^d \left[ \int_{\mu(y)}^{\nu(y)} f(x,y) dx \right] dy .$$

These are called the **iterated integrals**, and provide the technique for evaluating double integrals.

For a general region, we try to break it up into a finite set of nonoverlapping pieces, each of which is either type 1 or type 2. Then we evaluate over each piece by the iterated integral, and add them all together. This uses part (a) of the following proposition, all of which can be easily verified as for integration in one variable.

**Proposition 17.3** If  $R$  consists of  $n$  nonoverlapping regions  $R_1, R_2, \dots, R_n$ , then

$$b) \int_R f dA = \int_{R_1} f dA + \dots + \int_{R_n} f dA.$$

$$c) \int_R (f + g) dA = \int_R f dA + \int_R g dA. \quad 3) \text{ If } C \text{ is a constant}$$

$$(17.58) \quad \int_R C f dA = C \int_R f dA.$$

e) If  $f \geq g$  then

$$(17.59) \quad \int_R f dA \geq \int_R g dA.$$

f) The **area** of the region  $R$  is

$$(17.60) \quad A(R) = \int_R dA.$$

g) If  $f$  is non-negative, the **volume** of the region lying over the region  $R$  in the  $xy$ -plane and under the surface  $z = f(x, y)$  is

$$(17.61) \quad \int_R f dA.$$

## §17.4. Integration in Other Coordinates

### Polar coordinates

Calculations of double integrals are often simplified by turning to appropriate coordinates. If, for example, the problem setup is suggestive of polar coordinates, the change can be made as follows. Cover the region with a grid this time made up of the curves  $r = \text{constant}$ ,  $\theta = \text{constant}$  (see figure 17.11). Then form the sum

$$(17.62) \quad \sum f(r, \theta) \Delta A,$$

where now  $\Delta A$  is the area of one of the figures cut out by this grid, and  $f$  is evaluated at a point in the grid. If the grid is very fine, we can take its area to be the product of the lengths of its sides, respectively  $\Delta r$  and  $r\Delta\theta$ . Using this approximation, in the limit, we get  $dA = r dr d\theta$ , so that

$$(17.63) \quad \int_R f dA = \int \int f(r, \theta) r dr d\theta,$$

which we now can evaluate by appropriate iteration. So, if the region  $R$  is of the form  $\alpha \leq \theta \leq \beta$ ,  $u(\theta) \leq r \leq v(\theta)$ , we calculate the double integral by the iterated integral:

$$(17.64) \quad \int_R f(r, \theta) dA = \int_{\alpha}^{\beta} \left[ \int_{u(\theta)}^{v(\theta)} f(r, \theta) r dr \right] d\theta.$$

Figure 17.11

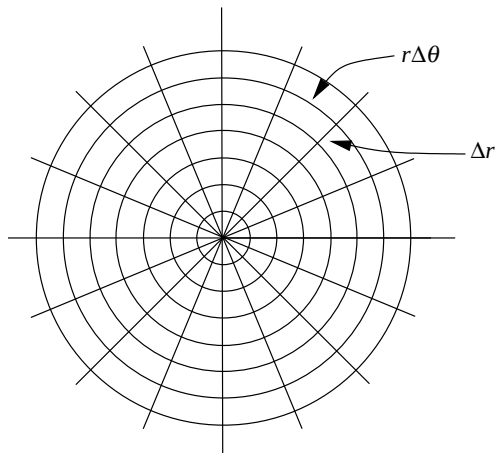
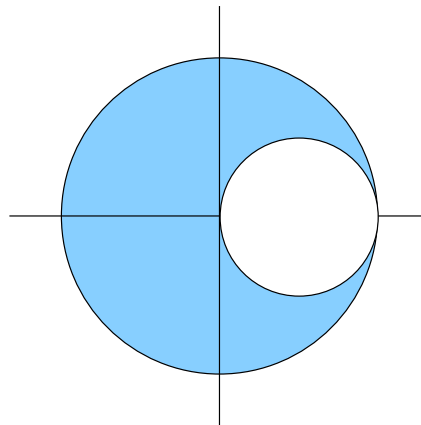


Figure 17.12



**Example 17.10** Find  $\iint_R x dA$  where  $R$  is the region bounded by the circle of radius 1 centered at the origin and the circle of radius  $1/2$  centered at the point  $(1/2, 0)$  (see figure 17.12).

If we move to polar coordinates we have  $x = r \cos \theta$ , the outer boundary of  $R$  is  $r = 1$ , and the inner boundary is given by  $r = \cos \theta$ . To proceed, we first notice that the integral is twice that of the integral over the part of  $R$  in the upper half plane, by symmetry. To reduce the integral to iterated integrals, we have to consider the pieces in the first and second quadrants separately. Denote these by  $I$  and  $II$  as in the figure.

$$(17.65) \quad \iint_I x dA = \int_0^{\pi/2} \int_{\cos \theta}^1 (r \cos \theta) r dr d\theta = \int_0^{\pi/2} \int_{\cos \theta}^1 r^2 \cos \theta dr d\theta.$$

The inner integral is

$$(17.66) \quad \frac{r^3}{3} \cos \theta \Big|_{\cos \theta}^1 = \frac{1}{3} (\cos \theta - \cos^4 \theta).$$

Using the double angle formula twice, we find

$$(17.67) \quad \cos^4 \theta = \frac{3}{8} + \frac{\cos(2\theta)}{2} + \frac{\cos(4\theta)}{4}.$$

Then

$$(17.68) \quad \iint_I x dA = \frac{1}{3} \int_0^{\pi/2} (\cos \theta - \cos^4 \theta) d\theta = \frac{1}{3} \left(1 - \frac{3\pi}{16}\right).$$

Now an easier computation gives

$$(17.69) \quad \iint_{II} x dA = \int_{\pi/2}^{\pi} \int_0^1 (r \cos \theta) r dr d\theta = -\frac{1}{3},$$

so that finally

$$(17.70) \quad \iint_R x dA = 2 \left( \frac{1}{3} \left(1 - \frac{3\pi}{16}\right) - \frac{1}{3} \right) = -\pi/8.$$

**Example 17.11** Find the center of mass of the region described in example 17.10 (see figure 17.12).

That region is bounded on the outside by a circle of radius 1, and on the inside by a circle of radius  $1/2$ , so has area  $\pi - \pi/4 = 3\pi/4$ . In example 17.6, we calculated the moment about  $x = 0$  to be

$$(17.71) \quad Mom_{x=0} = \int \int_R x dA = -\frac{\pi}{8},$$

so the  $x$  coordinate of the center of mass is  $x = -(\pi/8)/(3\pi/4) = -1/6$ . Since the region is symmetric about the  $x$ -axis, the center of mass is on that axis, and thus is at  $(-1/6, 0)$ .

Our next example is a trick calculation, obtained by working backwards from the iterated integral to the double integral.

**Example 17.12**  $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ .

Observe first that

$$(17.72) \quad \left( \int_0^\infty e^{-x^2} dx \right)^2 = \left( \int_0^\infty e^{-x^2} dx \right) \left( \int_0^\infty e^{-y^2} dy \right),$$

just by renaming the variable of integration in the second factor. But now, this last can be viewed as an iterated integral, and then as a double integral:

$$(17.73) \quad \left( \int_0^\infty e^{-x^2} dx \right) \left( \int_0^\infty e^{-y^2} dy \right) = \int_0^\infty \left[ \int_0^\infty e^{-(x^2+y^2)} dy \right] dx = \int \int_Q e^{-(x^2+y^2)} dA,$$

where  $Q$  is the first quadrant. Now, this is the integral of  $e^{-r^2}$  over  $Q$ , which in polar coordinates is given by  $0 \leq \theta \leq \pi/2$ ,  $0 \leq r \leq \infty$ . Making the change to polar coordinates, we get :

$$(17.74) \quad \left( \int_0^\infty e^{-x^2} dx \right)^2 = \int_0^{\pi/2} \left[ \int_0^\infty e^{-r^2} r dr \right] d\theta = \frac{\pi}{2} \left( \frac{1}{2} \right) = \frac{\pi}{4}.$$

Taking square roots, we get the result.

#### §17.4.1 General Coordinate Changes

Let  $\mathbf{L}$  and  $\mathbf{M}$  be two vectors in the plane which are not parallel, so that the area of the parallelogram they span,  $|\det(\mathbf{L}, \mathbf{M})|$ , is not zero. We can cover the plane with a grid of lines parallel to  $\mathbf{L}$  and  $\mathbf{M}$  (see figure 17.13), so that any point  $\mathbf{X}$  in the plane can be represented by a vector sum of the form  $u\mathbf{L} + v\mathbf{M}$ . We will call  $u, v$  the coordinates of the point relative to the basis  $\mathbf{L}, \mathbf{M}$ . Now if we take the grid sufficiently fine, we can approximate the area of a region  $R$  by the sum of the areas of the grid parallelograms wholly contained inside the region; that is, the limit of these sums, as the grid becomes infinitely fine, is the area of the region. If the side lengths of a typical parallelogram (in  $u, v$  coordinates) are  $\Delta u, \Delta v$ , the area of the parallelogram is  $|\det(\mathbf{L}, \mathbf{M})| \Delta u \Delta v$  (see figure 17.14). Thus, in the limit:

$$(17.75) \quad \int \int_R dA = \lim \sum |\det(\mathbf{L}, \mathbf{M})| \Delta u \Delta v = \int \int_S |\det(\mathbf{L}, \mathbf{M})| du dv.$$

where  $S$  is the region in  $u, v$  coordinates corresponding to  $R$ . Thus the area of  $R$  is  $|\det(\mathbf{L}, \mathbf{M})|$  times the area of  $S$ . This same argument works for the integral of a function over the region  $R$ :

$$(17.76) \quad \iint_R f(x, y) dA = \lim \sum f(\bar{x}, \bar{y}) |\det(\mathbf{L}, \mathbf{M})| \Delta u \Delta v$$

$$(17.77) \quad = \iint_S f(x(u, v), y(u, v)) |\det(\mathbf{L}, \mathbf{M})| du dv.$$

Figure 17.13

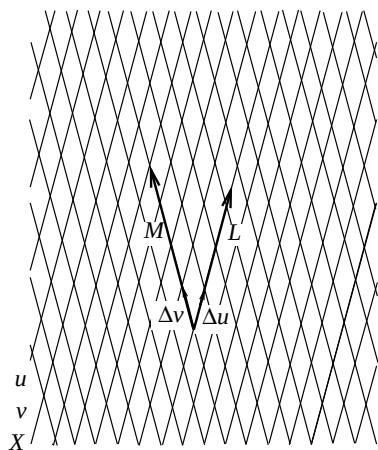
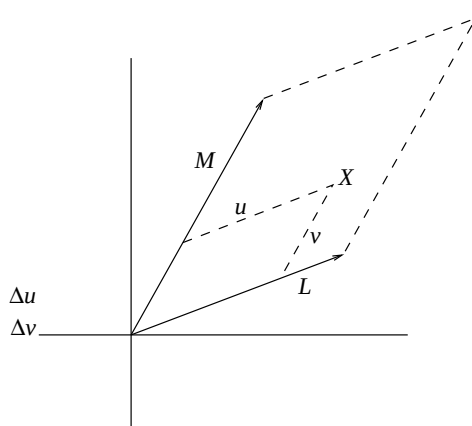


Figure 17.14



**Example 17.13** Find  $\int \int_R x dA$ , where  $R$  is the parallelogram with vertices at  $(0,0)$ ,  $(5,3)$ ,  $(2,6)$ ,  $(8,9)$ .

$R$  can be described as parallelogram with sides  $\mathbf{L} = 5\mathbf{i} + 3\mathbf{j}$ ,  $\mathbf{M} = 2\mathbf{i} + 6\mathbf{j}$ . We see (using figure 17.14) that a point  $(x, y)$  is in  $R$  if we can write the vector  $\mathbf{X} = x\mathbf{i} + y\mathbf{j} = u\mathbf{L} + v\mathbf{M}$  with  $0 \leq u \leq 1$ ,  $0 \leq v \leq 1$ . More precisely,  $x\mathbf{i} + y\mathbf{j} = (5u + 2v)\mathbf{i} + (3u + 6v)\mathbf{j}$ , which amounts to the change of coordinates

$$(17.78) \quad x = 5u + 2v \quad y = 3u + 6v$$

realizes  $R$  as the image of the unit square  $S$  in  $(u, v)$  space. We have

$$(17.79) \quad \det(\mathbf{L}, \mathbf{M}) = \begin{vmatrix} 5 & 3 \\ 2 & 6 \end{vmatrix} = 24.$$

Then

$$(17.80) \quad \iint_R x dA = \iint_S (3u + 6v) |\det(\mathbf{L}, \mathbf{M})| du dv = 24 \int_0^1 \left[ \int_0^1 (3u + 6v) du \right] dv = 4.5.$$

Now, suppose that we make a (not necessarily linear) change of variables in a region  $R$ :  $x = x(u, v)$ ,  $y = y(u, v)$ . To say that this is a change of variables is to say that the values of  $u$  and  $v$  are determined by the point  $(x, y)$  in  $R$ , that is, in principle, we can solve these equations for  $u$  and  $v$  in terms of  $x$  and  $y$ .

In terms of vectors we can write this as  $\mathbf{X}(u, v) = x(u, v)\mathbf{I} + y(u, v)\mathbf{J}$ . We can see how to calculate integrals in the  $(u, v)$  plane by following the above argument for a linear change of variables. However, now the factor by which we multiply is not constant, and depends at each point upon the relationship between differential rectangles at that point. This relationship is given by the linear approximation at that point. Specifically, If we select, at a point in  $(u, v)$  space, a rectangle with sides in the coordinate directions, and of side lengths  $du, dv$ , the image is, in the linear approximation, the parallelogram spanned by the vectors  $\mathbf{X}_u du$  and  $\mathbf{X}_v dv$  (see figures 17.15 and 17.16 ).

Figure 17.15

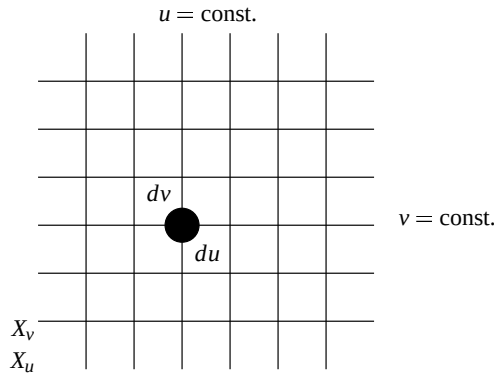
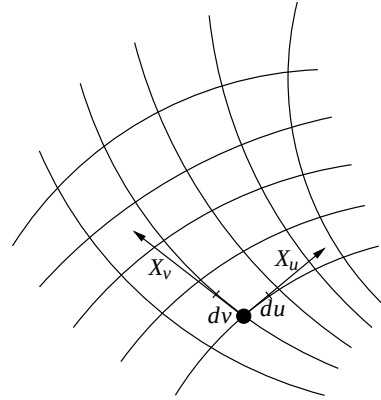


Figure 17.16



In this figure, the vectors  $\mathbf{X}_u$  and  $\mathbf{X}_v$  play the role of the vectors  $\mathbf{L}$  and  $\mathbf{M}$  in the linear case, since they represent a move of one unit in the respective coordinate directions. This parallelogram then has area  $|\det(\mathbf{X}_u, \mathbf{X}_v)|dudv = |\mathbf{X}_u du \times \mathbf{X}_v dv|$ .

**Definition 17.4** Given a change of coordinates  $x = x(u, v)$ ,  $y = y(u, v)$ , the determinant

$$(17.81) \quad \det(\mathbf{X}_u, \mathbf{X}_v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is called the **Jacobian** of the change of variables and is denoted by

$$(17.82) \quad \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}$$

**Proposition 17.4** If  $x = x(u, v)$ ,  $y = y(u, v)$  is a change of coordinates in the region  $R$ , then we have this equation for the differential of area:

$$(17.83) \quad dA = |\mathbf{X}_u \times \mathbf{X}_v|dudv = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right| dudv .$$

If  $R$  is the image of the region  $S$  in  $(u, v)$ -space, and  $f$  is an integrable function on  $R$ , then

$$(17.84) \quad \int_R f(x, y) dx dy = \int_S f(x(u, v), y(u, v)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right| dudv .$$



As we have seen in Chapter 13, the determinant  $\det(\mathbf{X}_u, \mathbf{X}_v)$  is the determinant of the two by two matrix whose rows are the components of the vectors; that is, the determinant in (17.83). (17.84) follows from (17.83) intuitively: we can cover the region  $R$  by a grid formed of the level curves of the functions  $u$  and  $v$ , and get a good approximation by the right hand side, for a grid fine enough.

To illustrate this, we return to polar coordinates as the change of variables  $x = r \cos \theta$ ,  $y = r \sin \theta$ , or, vectorially

$$(17.85) \quad \mathbf{X}(r, \theta) = r \cos \theta \mathbf{I} + r \sin \theta \mathbf{J} .$$

Now, for a differential rectangle in  $r, \theta$ -space of side lengths  $dr$ ,  $d\theta$ , the corresponding figure in  $x, y$ -space is the parallelogram bounded by the vectors  $\mathbf{X}_r dr$  and  $\mathbf{X}_\theta d\theta$  (see figure 17.11). We have

$$(17.86) \quad \mathbf{X}_r \times \mathbf{X}_\theta = \det \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} = r ,$$

and the element of area is

$$(17.87) \quad dA = |\mathbf{X}_r dr \times \mathbf{X}_\theta d\theta| = r dr d\theta .$$

**Example 17.14** Show that the area of an ellipse of major radius  $a$  and minor radius  $b$  is  $\pi ab$ .

We consider the ellipse as the region  $E$  bounded by the curve

$$(17.88) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

The change of variables  $x = au$ ,  $y = bv$  realizes this ellipse as the image of the unit disk  $D$  in  $(u, v)$  space. Thus

$$(17.89) \quad \text{Area} = \iint_E dx dy = \iint_D \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = ab \iint_D dx dy = \pi ab ,$$

since the last integral is the area of the disc of radius 1, which is  $\pi$ .

**Example 17.15** Find the area of the ellipse  $E$  bounded by the curve  $x^2 + 4xy + 13y^2 = 16$ .

If we complete the square the equation becomes

$$(17.90) \quad (x + 2y)^2 + 9y^2 = 16 .$$

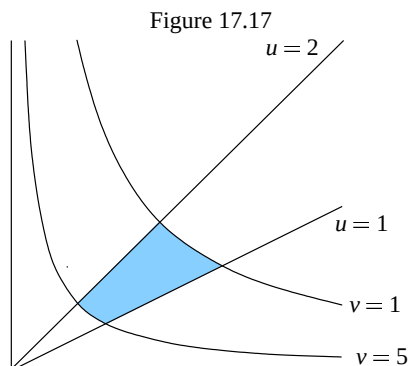
If we let  $u = x + 2y$ ,  $v = 3y$ , this is the image of the disk  $D$  in  $(u, v)$  space bounded by  $u^2 + v^2 = 16$ . The radius is 4, and the area is  $16\pi$ . Solving for  $x, y$  in terms of  $u, v$ , we have  $x = u - 2v/3$ ,  $y = v/3$ . The Jacobian of the change of variables is

$$(17.91) \quad \det \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3} .$$

Thus

$$(17.92) \quad \text{Area}(E) = \iint_E dx dy = \frac{1}{3} \iint_D du dv = \frac{1}{3} (\text{Area}(D)) = \frac{16\pi}{3} .$$

**Example 17.16** Find the area of the region  $R$  in the first quadrant bounded by the curves  $x = y$ ,  $x = 2y$ ,  $xy = 1$ ,  $xy = 5$ .



Make the change of variables  $u = x/y$ ,  $v = xy$ . Then the region can be described by the inequalities  $1 \leq u \leq 2$ ,  $1 \leq v \leq 5$ . Let  $S$  represent this region in  $u, v$  space. To integrate in  $u, v$  space, we have to find the Jacobian. First we solve the equations for  $x$  and  $y$  in terms of  $u$  and  $v$ , obtaining:

$$(17.93) \quad x = \sqrt{uv}, \quad y = \sqrt{\frac{v}{u}},$$

so that

$$(17.94) \quad \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & -\frac{1}{2u}\sqrt{\frac{v}{u}} \\ \frac{1}{2}\sqrt{\frac{u}{v}} & \frac{1}{2}\sqrt{\frac{1}{uv}} \end{pmatrix} = \frac{1}{4}\sqrt{\frac{v}{u^2v}} + \frac{1}{4u}\sqrt{\frac{vu}{uv}} = \frac{1}{2u}.$$

Thus

$$(17.95) \quad \text{Area}(R) = \iint_R dx dy = \iint_S \frac{1}{2u} du dv = \frac{1}{2} \int_1^2 \left[ \int_1^5 \frac{1}{u} dv \right] du = 2 \int_1^2 \frac{du}{u} = 2 \ln 2.$$

### §17.4.2 Surface Area

Consider a surface  $\mathbf{S}$  in three dimensions given parametrically by

$$(17.96) \quad \mathbf{X}(u, v) = x(u, v)\mathbf{I} + y(u, v)\mathbf{J} + z(u, v)\mathbf{K}$$

where  $(u, v)$  lies in a region  $R$ . To approximate the area of  $\mathbf{S}$  we cut  $R$  up into small rectangles given by a fine grid, calculate the area of the parallelogram (in the tangent plane at a point in  $R$ ) approximating the image of the rectangle on  $\mathbf{S}$ , and add this up over all rectangles, as we did for the plane (see figures 17.15 and 17.16; now the right hand figure is a grid on a surface in space). Turning to the language of differentials, let  $dS$  represent the differential of the surface area. This is the area of the rectangle in space spanned by the vectors  $\mathbf{X}_u du$  and  $\mathbf{X}_v dv$ . Thus

$$(17.97) \quad dS = |\mathbf{X}_u du \times \mathbf{X}_v dv|,$$

and thus the area is

$$(17.98) \quad \text{Area}(\mathbf{S}) = \int \int_R dS = \int \int_R |\mathbf{X}_u \times \mathbf{X}_v| du dv \quad (\text{Surface given parametrically}).$$

**Example 17.17** Find the area of the piece of the paraboloid  $z = x^2 + y^2$  cut off by the plane  $z = 1$ .

Parametrize the surface using cylindrical coordinates:

$$(17.99) \quad \mathbf{X}(x, y) = r \cos \theta \mathbf{I} + r \sin \theta \mathbf{J} + r^2 \mathbf{K}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1.$$

Then

$$(17.100) \quad \mathbf{X}_r = \cos \theta \mathbf{I} + \sin \theta \mathbf{J} + 2r \mathbf{K}, \quad \mathbf{X}_\theta = -r \sin \theta \mathbf{I} + r \cos \theta \mathbf{J}.$$

$$(17.101) \quad dS = |\mathbf{X}_r \times \mathbf{X}_\theta| dr d\theta = r \sqrt{1 + 4r^2} dr.$$

$$(17.102) \quad \text{Area} = \int_0^{2\pi} \int_0^1 r \sqrt{1 + 4r^2} dr d\theta = \frac{\pi}{6} (\sqrt{125} - 1).$$

If the surface is the graph of a function  $z = f(x, y)$ , we can look at this as given parametrically by

$$(17.103) \quad \mathbf{X}(x, y) = x \mathbf{I} + y \mathbf{J} + f(x, y) \mathbf{K},$$

and we find  $\mathbf{X}_x = \mathbf{I} + f_x \mathbf{K}$ ,  $\mathbf{X}_y = \mathbf{J} + f_y \mathbf{K}$  and so, from (17.97):  $dS = \sqrt{1 + f_x^2 + f_y^2} dx dy$  and we have

$$(17.104) \quad \text{Area}(\mathbf{S}) = \int \int_R dS = \int \int_R \sqrt{1 + f_x^2 + f_y^2} du dv \quad (\text{Surface given as a graph}).$$

**Example 17.18** Find the area of the cone  $z^2 = x^2 + y^2$  lying over the triangle in the first quadrant bounded by the line  $2x + y = 2$ .

We represent the surface as the graph of the function  $f(x, y) = \sqrt{x^2 + y^2}$  over the region  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2 - 2x$ . Differentiating:

$$(17.105) \quad f_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2}},$$

so

$$(17.106) \quad dS = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} dx dy = \sqrt{2} dx dy.$$

Thus

$$(17.107) \quad \text{Area} = \sqrt{2} \int_0^1 \int_0^{2-2x} dy dx = \sqrt{2}.$$

**Example 17.19** If the function is given in polar coordinates:  $z = f(r, \theta)$ , we write

$$(17.108) \quad \mathbf{X}(r, \theta) = r \cos \theta \mathbf{I} + r \sin \theta \mathbf{J} + f(r, \theta) \mathbf{K}.$$

Then  $\mathbf{X}_r = \cos \theta \mathbf{I} + \sin \theta \mathbf{J} + f_r \mathbf{K}$ ,  $\mathbf{X}_\theta = -r \sin \theta \mathbf{I} + r \cos \theta \mathbf{J} + f_\theta \mathbf{K}$  and the calculation gives

$$(17.109) \quad dS = \sqrt{r^2 + r^2 f_r^2 + f_\theta^2} dr d\theta.$$

Notice, if the surface is the plane  $z = 0$ , then  $dS$  is just the element of area in polar coordinates:  $r dr d\theta$ .