Change of Coordinates in Two Dimensions

Suppose that E is an ellipse centered at the origin. If the major and minor axes are horizontal and vertical, as in figure 15.1, then the equation of the ellipse is

(15.1)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \, ,$$

where a and b are the lengths of the major and minor radii.

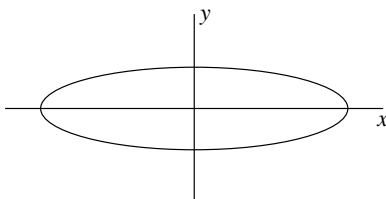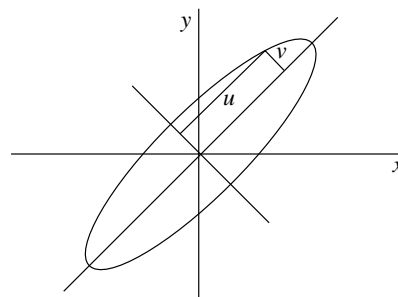Figure 15.1   Figure 15.2

However, if the axes of E are neither horizontal nor vertical, as in figure 15.2, then we do not have this simple form of the equation. What we do know, since the curves in figures 15.1 and 15.2 are the same but for a rotation, is this: for u and v as shown in figure 15.2,

(15.2)
$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1 \, .$$

Now, the point here is that u and v can be expressed in terms of the cartesian coordinates, and in turn, x and y can be determined from u and v. This replacement of one pair of variables which determine a point by another is called a **change of coordinates**. We now show how to do this in the context of the ellipse of figure 2. First, recall Proposition 13.3, and in particular example 13.6. Introduce a unit vector $\mathbf{L}$ in the positive direction of the major axis, so that $\mathbf{L}^\perp$ is in the direction of the minor axis. Now, any point X can be represented by the vector $\mathbf{X} = u\mathbf{L} + v\mathbf{L}^\perp$ (see figure 15.3). The following proposition tells us how to find u and v.
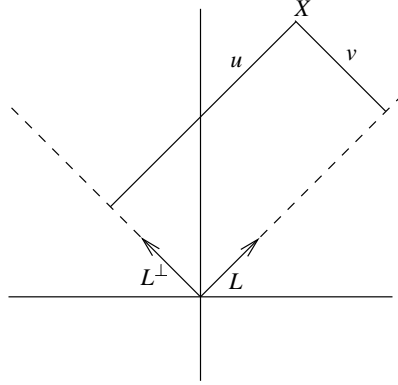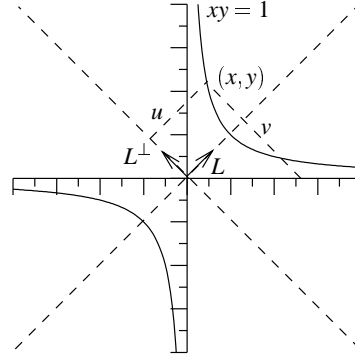
Figure 15.3

Figure 15.4

Proposition 15.1 *Let* **L** *be a unit vector in the plane. Then any vector* **X** *can be written as*

(15.3)
$$\mathbf{X} = u\mathbf{L} + v\mathbf{L}^{\perp} \quad \text{where} \quad u = \mathbf{X} \cdot \mathbf{L}, \quad v = \mathbf{X} \cdot \mathbf{L}^{\perp}.$$

To see this, we calculate the dot products;

(15.4)
$$\mathbf{X} \cdot \mathbf{L} = (u\mathbf{L} + v\mathbf{L}^{\perp}) \cdot \mathbf{L} = u\mathbf{L} \cdot \mathbf{L} + v\mathbf{L}^{\perp} \cdot \mathbf{L} = u,$$

(15.5)
$$\mathbf{X} \cdot \mathbf{L}^{\perp} = (u\mathbf{L} + v\mathbf{L}^{\perp}) \cdot \mathbf{L}^{\perp} = u\mathbf{L} \cdot \mathbf{L}^{\perp} + v\mathbf{L}^{\perp} \cdot \mathbf{L}^{\perp} = v,$$

since $\mathbf{L} \cdot \mathbf{L} = 1$, $\mathbf{L}^{\perp} \cdot \mathbf{L}^{\perp} = 1$, $\mathbf{L} \cdot \mathbf{L}^{\perp} = 0$. In this way, if we are given a geometric description of the ellipse, we can find its equation in the cartesian coordinates x, y.

Example 15.1 Let E be the ellipse centered at the origin, with major radius of length 5, major axis the line $3x - 4y = 0$, and minor radius of length 2. Find the equation of the ellipse.

The point $(4,3)$ is on the given line, so $4\mathbf{I} + 3\mathbf{J}$ lies in the direction of the major axis. The length of this vector is 5, so we can take the unit vector in the direction of the major axis to be $\mathbf{L} = (4\mathbf{I} + 3\mathbf{J})/5$. Thus, in the context of the above discussion,

(15.6)
$$\mathbf{L} = \frac{4}{5}\mathbf{I} + \frac{3}{5}\mathbf{J}, \quad \mathbf{L}^{\perp} = -\frac{3}{5}\mathbf{I} + \frac{4}{5}\mathbf{J},$$

and a point $\mathbf{X} = x\mathbf{I} + y\mathbf{J}$ is on the ellipse if and only if

(15.7)
$$\frac{u^2}{5^2} + \frac{v^2}{2^2} = 1 \quad \text{where} \quad u = \mathbf{X} \cdot \mathbf{L}, \quad v = \mathbf{X} \cdot \mathbf{L}^{\perp}.$$

Now, $\mathbf{X} \cdot \mathbf{L} = (4/5)x + (3/5)y$, $\mathbf{X} \cdot \mathbf{L}^{\perp} = -(3/5)x + (4/5)y$, so the equation of the ellipse is

(15.8)
$$\frac{[(4/5)x + (3/5)y]^2}{25} + \frac{[-(3/5)x + (4/5)y]^2}{4} = 1.$$

This simplifies to $.1156x^2 - .2016xy + .1744y^2 = 1$, or $289x^2 - 504xy + 436y^2 = 2500$.

In general, it happens that, in solving a particular problem, the situation can be easily realized in variables adapted to the problem, but the solution requires presentation in terms of an initial cartesian coordinate system. For example, the above ellipse is easily described in terms of the variables u and v adapted to its axes, but to realize the ellipse by an equation, we had to represent u and v in terms of x and y. We now state the proposition which gives the procedure used in example 15.1.

Proposition 15.2 *Given a unit vector* $\mathbf{L} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$ *we can write any vector* $\mathbf{X} = x\mathbf{I} + y\mathbf{J}$ *as* $\mathbf{X} = u\mathbf{L} + v\mathbf{L}^\perp$, *where* $u = \mathbf{X}\cdot\mathbf{L}$ *and* $v = \mathbf{X}\cdot\mathbf{L}^\perp$; *that is*

(15.9) $u = x\cos\theta + y\sin\theta , \quad v = -x\sin\theta + y\cos\theta .$

We refer to equations (15.9) as a **change of coordinate** by **rotation** through the angle θ. We can also reverse the roles of these variables and return to x, y from u, v just by rotating back through an angle $-\theta$. This gives us the equations

(15.10) $x = u\cos\theta - v\sin\theta , \quad y = u\sin\theta + v\cos\theta .$

Of course, this is just what we get by solving equations (15.9) for x and y in terms of u and v.

Example 15.2 The curve $xy = 1$ is symmetric about the axes $x = y$, $x = -y$. Write the curve in coordinates u, v relative to these axes, as in figure 15.4.

Since the line $x = y$ makes an angle of $45°$ with the horizontal, the change of coordinates is accomplished by a rotation through $45°$. Since $\cos(45°) = \sin(45°) = 1/\sqrt{2}$, we have the relations

(15.11) $u = \dfrac{x+y}{\sqrt{2}} , \quad v = \dfrac{-x+y}{\sqrt{2}} , \quad x = \dfrac{u-v}{\sqrt{2}} , \quad y = \dfrac{u+v}{\sqrt{2}} .$

Substituting for x and y in terms of u and v in $xy = 1$, we get

(15.12) $xy = \left[\dfrac{u-v}{\sqrt{2}}\right]\left[\dfrac{u+v}{\sqrt{2}}\right] = 1 \quad \text{leading to} \quad u^2 - v^2 = 2 ,$

the equation of a hyperbola in the u, v coordinates.

As we have seen in the above examples, a hyperbola or ellipse leads to a quadratic equation, which will have a nonzero xy term if the axes are not horizontal and vertical. This is always true; as well as the reverse: any quadratic equation is the equation of a conic curve. We now see how to find the standard description of the conic from the equation (first with an example).

Example 15.3 Let C be the curve given by the equation

(15.13) $x^2 - xy = 12 .$

Find coordinates which put this in standard form.

We want to make a substitution of the form (15.10) so that the coefficient of the uv term is 0. Making the substitution gives us

(15.14) $(u\cos\theta - v\sin\theta)^2 - (u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta) = 12 .$

The coefficient of *uv* is

(15.15)　　　　　　　　$-2\cos\theta\sin\theta - \cos^2\theta + 2\sin^2\theta \quad$ or $\quad -\sin(2\theta) - \cos(2\theta)$.

This is zero when $\tan(2\theta) = -1$, or $\theta = -\pi/8$. For this value of θ we now compute the coefficients A and B of u^2 and v^2:

(15.16)　　　　　　$A = \cos^2\theta - \cos\theta\sin\theta = \frac{1}{2}(1 + \cos(2\theta) - \sin(2\theta)) = \frac{1 + \sqrt{2}}{2}$,

(15.17)　　　　　　$B = \sin^2\theta + \cos\theta\sin\theta = \frac{1}{2}(1 - \cos(2\theta) + \sin(2\theta)) = \frac{1 - \sqrt{2}}{2}$,

so that the equation for the curve in the u, v coordinates is

(15.18)　　　　　　　　　　　$(\sqrt{2} + 1)u^2 - (\sqrt{2} - 1)v^2 = 24$,

the equation of a hyperbola.

　　Following this example, given any quadratic equation in *x* and *y*:

(15.19)　　　　　　　　　　　$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

we can find a rotation which eliminates the cross term. The resulting equation in the new variables u, v is that of a conic section. Of course, there will be exceptional cases; for example the equation $x^2 + y^2 + 1 = 0$ has no solutions, and the equation $x^2 - y^2 = 0$ is a pair of straight lines. But, if (15.19) defines a curve, it must be an ellipse, hyperbola or parabola. If we introduce the new variables u and v by a rotation through an angle θ, the equation in the new coordinates is still quadratic in u and v; that is, the equation is of the form

(15.20)　　　　　　　　　　　$A'u^2 + B'uv + C'v^2 + D'u + E'v + F' = 0$

where the new coefficients are expressed in terms of θ and the old ones. By setting $B' = 0$, we see how to choose θ. So, let's make the substitution (15.10) in the equation (15.20). The part which is purely quadratic is

(15.21)　　　$A(u\cos\theta - v\sin\theta)^2 + B(u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta) + C(u\sin\theta + v\cos\theta)^2$.

The coefficient of *uv* in this expression is

(15.22)　　　　　　　$B' = -2A\cos\theta\sin\theta + B(\cos^2\theta - \sin^2\theta) + 2C\cos\theta\sin\theta$.

Set this to zero and solve for θ. Using double angle formulas, the equation is

(15.23)　　　　　　$(-A + C)\sin(2\theta) + B\cos(2\theta) = 0, \quad$ or $\quad \tan(2\theta) = \frac{B}{A - C}$.

If $A = C$, the denominator is zero, so we take $2\theta = \pi/2$, or $\theta = \pi/4$. Now, the equation (15.20) (with $B' = 0$) is of the form considered in chapter 11, and can be put in standard form with center at some other point.

Example 15.4   Let C be the curve given by the equation

(15.24) $$x^2 - 2\sqrt{3}xy - 3y^2 + 6\sqrt{3}x + 6y = 16 \,.$$

Find coordinates which put C in standard form.

First, we use (15.23) to find the angle of rotation:

(15.25) $$\tan(2\theta) = \frac{-2\sqrt{3}}{1-3} = \sqrt{3} \,, \quad \text{so} \quad 2\theta = \frac{\pi}{3} \,, \quad \theta = \frac{\pi}{6} \,,$$

and $(\sin\theta = 1/2, \ \cos\theta = \sqrt{3}/2)$, the substitution (15.10) is

(15.26) $$x = \frac{\sqrt{3}u - v}{2} \,, \quad y = \frac{u + \sqrt{3}}{2} \,.$$

We do this in two steps. First, the quadratic terms of (15.24) are, in the coordinates u, v:

(15.27) $$\frac{1}{4}[(3u^2 - 2\sqrt{3}uv + v^2) - 2\sqrt{3}(\sqrt{3}u^2 + 2uv + \sqrt{3}v^2) + 3(u^2 + 2\sqrt{3}uv + 3v^2]$$

which reduces to v^2. Now incorporate the linear terms of (15.24) in terms of u, v:

(15.28) $$v^2 + \frac{1}{2}\left[6\sqrt{3}(\sqrt{3}u - v) + 6(u + \sqrt{3}v)\right] = 16 \,,$$

which can be put in the standard form

(15.29) $$v^2 = -\frac{15}{2}\left(u - \frac{32}{15}\right) \,.$$

Thus the curve is a parabola, with axis at an angle of $\pi/3$ with the x-axis, which opens downward.

We summarize this discussion as follows.

Proposition 15.3 *A curve given by the equation*

(15.30) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is a conic section. Rotate coordinates by the angle θ given by

(15.31) $$\tan(2\theta) = \frac{B}{A-C} \,;$$

that is, make the substitution in (15.38):

(15.32) $$x = u\cos\theta - v\sin\theta \,, \quad y = u\sin\theta + v\cos\theta \,.$$

There is no uv term, so after completing the squares, the equation is in standard form. In particular, the axes of the conic are at an angle θ with the coordinate axes.

Proposition 15.4 *For a curve given by equation (15.38),*
If $B^2 - 4AC < 0$, the curve is an ellipse.
If $B^2 - 4AC > 0$, the curve is a hyperbola.

If $B^2 - 4AC = 0$, *the curve is a parabola.*

To indicate why this is true, let us consider just the quadratic terms and start with an equation of the form

(15.33) $$Ax^2 + Bxy + Cy^2 = 1 .$$

Supposing $A > 0$, we complete the square for the first two terms, rewriting (15.49) as

(15.34) $$A\left(x^2 + 2\frac{By}{2A}x +]\left(\frac{By}{2A}\right)^2\right) - \frac{B^2}{2A}y^2 + Cy^2 = 1 ,$$

or

(15.35) $$A\left(x + \frac{By}{2A}y\right)^2 + \frac{4AC - B^2}{4A}y^2 = 1 .$$

If the coefficients of the squared terms are both positive, then there are no solutions for large x and y, so the curve is an ellipse. On the other hand, if the signs of the coefficients are different, there are always solutions for large x and y, so the curve must be a hyperbola. Thus the shape is determined by the sign of $4AC - B^2$, and if we carefully follow through the argument, we arrive at proposition 15.4.

Example 15.5   Describe the curve $x^2 + xy + y^2 = 1$.
    Since $B^2 - 4AC = -1 < 0$, this is an ellipse. Since $A = C$, we need to rotate coordinates by $\pi/4$. We make the subsititution

(15.36) $$x = \frac{u - v}{\sqrt{2}} , \quad y = \frac{u + v}{\sqrt{2}}$$

getting

(15.37) $$\frac{u^2 - 2uv + v^2}{2} + \frac{u^2 - v^2}{2} + \frac{u^2 + 2uv + v^2}{2} = 1 ,$$

which reduces to $3u^2 + 2v^2 = 2$.

§15.1. Special Coordinate Systems

Often a problem can be seen as that of understanding the motion of a particle relative to a fixed point or a fixed axis. In these cases it is useful to express everything in coordinates which emphasize positions relative to the fixed point or axis.

§15.1.1 Polar coordinates

First, we recall, from Chapter 11, polar coordinates in the plane. We consider the fixed point as the origin of these coordinates, and take the positive x-axis as the "zero" direction. Then any other direction is described by the angle between it and the positive x axis, which we denote as θ. The distance of a

point on this line from the origin is denoted r. These equations relate the cartesian coordinates (x, y) with the polar coordinates r, θ:

$$(15.38) \qquad x = r\cos\theta\ , \quad y = r\sin\theta\ , \qquad r = \sqrt{x^2 + y^2}\ , \quad \theta = \arctan\frac{y}{x}$$

Polar coordinates have some ambiguities. Every value of (r, θ) determines a point in the plane. However, if $r = 0$, the point is the origin, and θ doesn't make sense. Secondly, the values (r, θ) and $(r, \theta + 2\pi)$, and in fact, $(r, \theta + 2n\pi)$ for any n give the same point. The curve $\theta = a$ is the ray of angle a emanating from the origin, and the curve $r = a$ is the circle of radius a centered at the origin (see figure 15.5).

In three dimensions, we introduce two new coordinate systems, the first oriented toward a fixed axis, the z-axis, and the second oriented toward the origin.

§15.1.2 Cylindrical coordinates

Here a point is described by its z-coordinate and its polar coordinates in the plane (see figure 15.6).
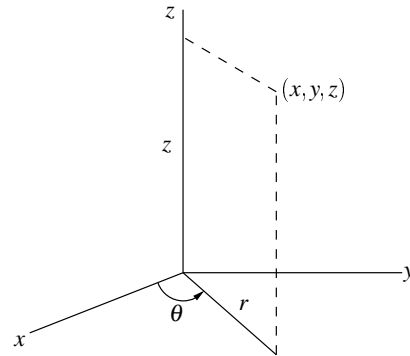
Figure 15.5
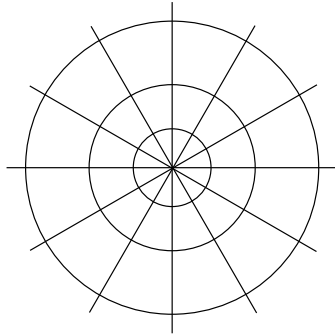
Figure 15.6

The formulas for the change from cartesian coordinates are

$$(15.39) \qquad x = r\cos\theta\ , \quad y = r\sin\theta\ , \quad z = z\ , \quad r = \sqrt{x^2 + y^2}\ , \quad \theta = \arctan\frac{y}{x}$$

The equation $r = at$ is a circular cylinder of radius a centered along $z = 0$; $\theta = a$ describes the half plane with its edge along $z = 0$ making an angle a with the xz-plane, and $z = a$ is a horizontal plane .
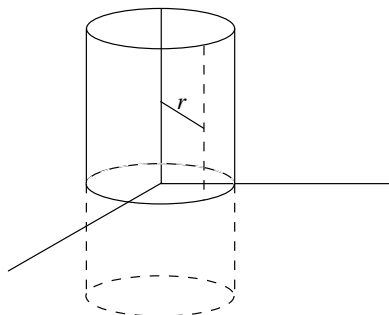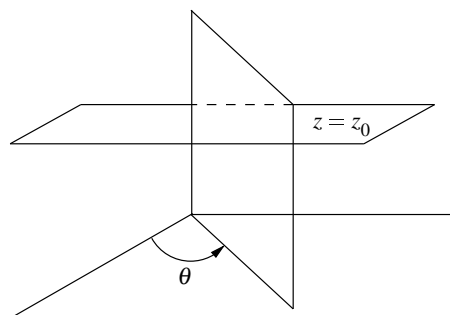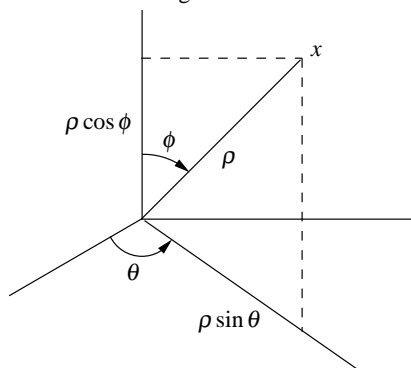
Figure 15.7

Figure 15.8

§15.1.3 Spherical coordinates

These coordinates are oriented toward the origin, so that a point is described by its distance ρ from the origin and the ray from the origin on which it lies. We describe the ray by the angle ϕ it makes with the z-axis and the angle θ it makes with the xz-plane (see figure 15.9).

Figure 15.9

We can read off from figure 15.9 the equations relating spherical coordinates with polar coordinates;

(15.40) $$x = \rho \sin \phi \cos \theta \, , \quad y = \rho \sin \phi \sin \theta \, , \quad z = \rho \cos \phi \, .$$

Note that, although θ ranges through a whole circle, ϕ ranges from 0 to π. The curve $\rho = a$ is the sphere of radius a centered at the origin, $\theta = a$ a half-plane, and $\phi = a$ the half-cone with axis the z-axis, making the angle a with its axis (see figures 15.10-15.12).

Example 15.6 Describe the curves $C_1 : \phi = \phi_0$, $\rho = 1$, $C_2 : \theta = \theta_0$, $\rho = R$. Give their equations in cylindrical and cartesian coordinates.

The curve C_1 is the intersection of the cone $\phi = \phi_0$ with the sphere of radius R. If we think of this sphere as the globe, C_1 is a circle of latitude. The radius of this circle is $R \sin \phi_0$, and its center is on

the z-axis, at a distance $R\cos\phi_0$ from the origin. In cylindrical coordinates, this curve lies on the plane $z = R\cos\phi_0$, and the cylinder $r = R\sin\phi_0$; these are then the equations of C_1 in cylindrical coordinates. In rectangular coordinates the equations are $x^2 + y^2 = (R\sin\phi_0)^2$, $z = R\cos\phi_0$.

The curve C_2 is the intersection of the plane $\theta = \theta_0$ with the sphere of radius R. If we think of this sphere as the globe, C_2 is a circle of longitude. Its center is the origin and its radius is R. In cylindrical coordinates, C_2 is given by the equations $r^2 + z^2 = R^2$, $\theta = \theta_0$. In cartesian coordinates, the equations are $x^2 + y^2 + z^2 = R^2$, $y = x\tan\theta_0$. (When $\theta_0 = \pi/2$ the second equation is $x = 0$.)
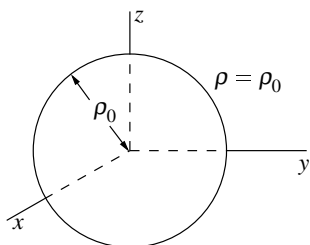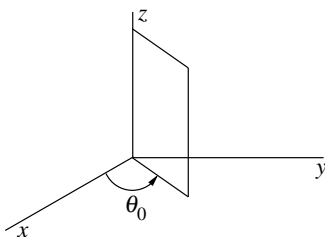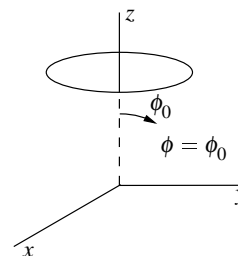
| Figure 15.10	Figure 15.11	Figure 15.12

§15.2. Surfaces; Graphs and Level curves

A relation among the variables x, y, z defines a **surface** in three dimensions: the set of all x, y, z which satisfy the equation. For example, we have seen that a linear relation $ax + by + cz + d = 0$ is the equation of a plane; that is, the set of all points (x, y, z) which satisfy that relation is a plane. Similarly, the sphere of radius R has the equation $x^2 + y^2 + z^2 = R^2$. As we have observed already, it is sometimes difficult to visualize a surface given by an equation in three variables. In this section we shall discuss various ways of visualizing surfaces.

In the case that the relation can be solved for z in terms of x and y, then the surface is the **graph** of the function $z = f(x, y)$. In this case it is a good idea to try to sweep out the surfaces by the curves of intersection of the curve with the planes $z = $ const. These are the **level curves** of the surface. Then we can imagine the surface as a stack of these level sets. In order to understand how the level sets stack, we may want to look at representative *profiles*: these are the curves of intersection of the surface with planes perpendicular to the xy-plane.

Example 15.7   Draw the level curves of the surface $z = 4 - x^2 - y^2$, and sketch the surface.

We see first of all, since $x^2 + y^2$ is never negative, that $z \leq 4$. The level surface $z = 4$ is just the origin $(0,0)$, but as z decreases from 4 we get a family of circles centered at the origin of ever increasing radius (the radius is $\sqrt{4 - z}$). Our surface then is a stack of circles. To see the shape of the stack, we look at a representative profile: the intersection of the surface with a plane through the z-axis. For example, for $y = 0$ we get the parabola $z = 4 - x^2$, and now can safely sketch the graph.

Figure 15.13
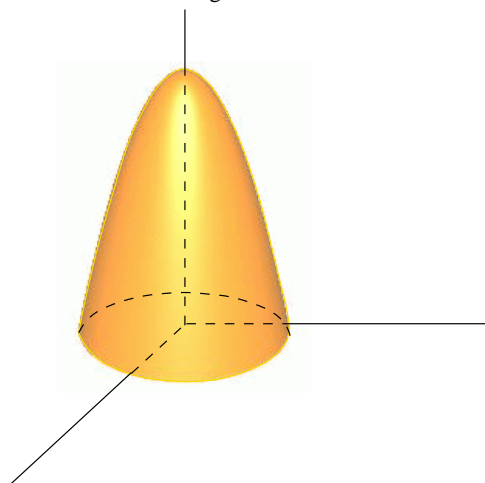
Example 15.8 Do the same for the surface $z = y - x^2$.

Setting z equal to the constant z_0, we get the parabola $x^2 = y - z_0$, so the level curves are the family of parabolas with axis the y-axis, opening upward, with vertex at $(0, z_0)$. We have shown typical level curves in figure 15.14. Thus the surface is is a stack of parallel parabolas with vertex moving linearly up the y-axis; that is, the vertices lie on the line $y = z$, $x = 0$. To get a further idea of the shape of the surface we look at a profile $y = $ constant , say $y = 0$. There the surface is given by the parabola $z = -x^2$, a parabola opening downward. Putting this information together we get figure 15.15.

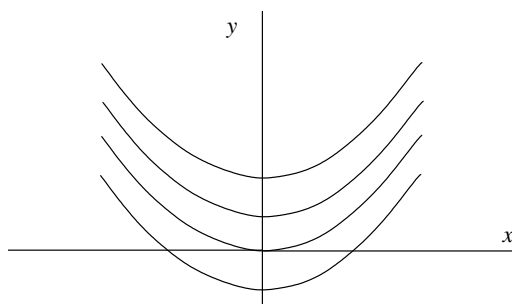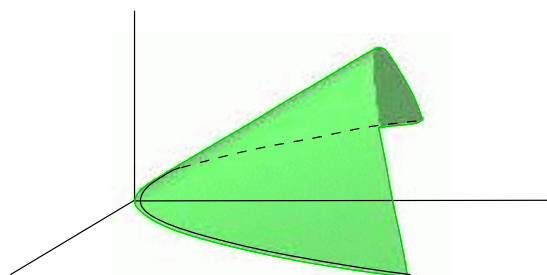Figure 15.14   Figure 15.15

As is clear from these examples, sketching surfaces is an imprecise science, and the configuration of level sets gives an idea of the shape, but not very precise. If we draw a large number of level sets on the xy plane, we can observe that at points where the level sets are close together, the surface is steep, and where they are far apart, the surface is quite flat. This is illustrated in figures 15.16 and 15.17: figure 15.16 is that of the surface, and figure 15.17 the configuration of its level sets in the xy-plane.
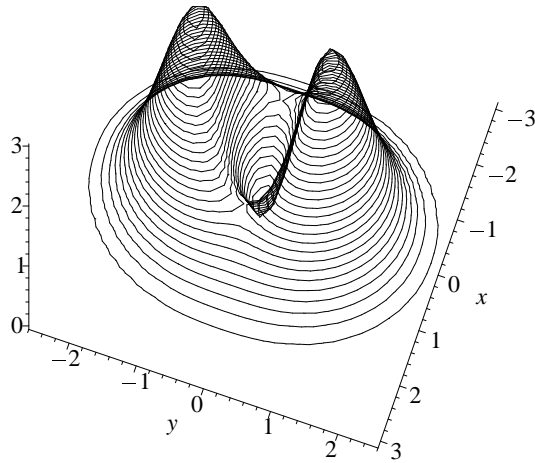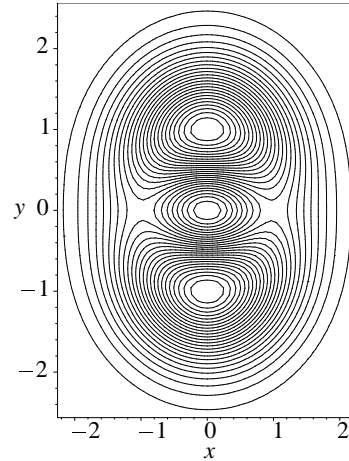
Figure 15.16

Figure 15.17

§15.3. Cylinders and Surfaces of Revolution

Starting with a curve C in a plane Π, the surface swept out by the translates of this curve in the direction perpendicular to Π is the **cylinder** over the curve C. This is the case when the relation defining the surface S is independent of one of the variables. For example, the surface S given by the relation $x^2 - y^2 = 1$ is independent of z, so if (x, y) is a point satisfying this relation, then all points (x, y, z) are on the surface, so S is the cylinder over the hyperbola $C : x^2 - y^2 = 1$ in the xy-plane.

Example 15.9   Sketch the surface $z = 9 - x^2$.

Since the relation is independent of x we just draw the parabola given by this equation on the xz-plane, and extend it by lines parallel to the y-axis.

Figure 15.18
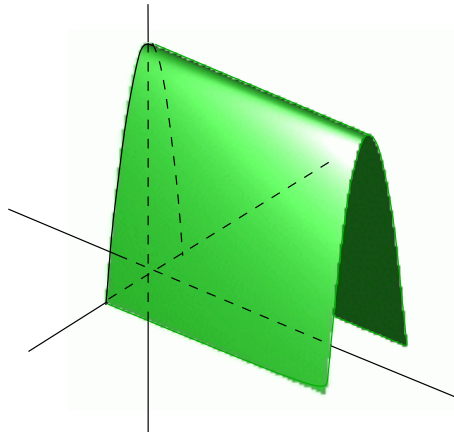
If a surface has the property that, for a particular line L, the intersection of the surface with a plane perpendicular to L is a circle centered on L, then the surface is a **surface of revolution** about the **axis** L. Suppose that S is a surface of revolution about the z axis. Then the intersection of S with the half-plane $y = 0$, $x > 0$ completely determines the surface. Let C be that curve of intersection, and suppose it is given by the equation $z = f(x)$. If $(x_0, 0, z_0)$ is a point on C, then every point on the plane $z = z_0$ whose distance from the z axis is x_0 is on S. That is, if $\sqrt{x^2 + y^2} = x_0$, then $z_0 = f(\sqrt{x^2 + y^2})$, so (x, y, z_0) is on the surface S. Thus the equation of a surface S of revolution is given by the equation $z = f(x)$ defining its profile just by replacing x by $\sqrt{x^2 + y^2}$: S is the surface $z = f(\sqrt{x^2 + y^2})$.

Example 15.10   Sketch the surface $z^2 = x^2 + y^2$.

This is the surface obtained by revolving the curve $z^2 = x^2$ about the z axis. Since that curve consists of the two lines $z = \pm x$, we get the cone in figure 15.24.

§15.4. Quadric surfaces

These are the surfaces which are given by a quadratic relation among the variables. By completing the square, and - if necessary - rotating the axes, we can reduce every quadric surface to one of the surfaces in this section. It is a good exercise to trace out these surfaces using the technique of level sets and profiles from the preceding sections. The figures are collected together at the end of the text. It is essential to become familiar with these surfaces, for they are the fundamental examples for the rest of the course.

Sphere of radius R $x^2 + y^2 + z^2 = R^2$ (Figure 15.19)
The sphere is symmetric about all axes and planes through its center. The intersection of the sphere with any plane is a circle. The intersection of the sphere with a plane through its center is a **great circle**.

Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (Figure 15.20)
This equation is just that of the sphere, but with the coordinates x, y, z replaced by x/a, y/b, z/c. The effect is that the sphere has been dilated in the $x-$, $y-$, $z-$directions by the factors a, b, c respectively, producing the ellipsoid with vertices $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0, \pm c)$ along the coordinate axes.

Hyperboloid of one sheet $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ (Figure 15.21)
First, let's consider the case $a = b = c = 1$. Then the equation can be written as

(15.41) $$x^2 + y^2 = 1 + z^2$$

so the intersection of this surface with the plane $z = z_0$ is a circle centered at the origin of radius $\sqrt{1 + z_0^2}$. This is a stack of circles of ever increasing radius as we move away from the xy-plane. If we set $y = 0$, we get the profile $x^2 - z^2 = 1$: a hyperbola, and thus figure 15.21. We could also have come to this figure by observing that this is the surface of revolution of this hyperbola.

Note that if we set $x = 1$, $y = z$ in equation (15.41) we get an identity. Thus this line lies on the surface. More importantly, since this is a surface of revolution, if we revolve this line about the z-axis, we generate the surface (see figure 15.22). The line $x = 1$, $y = -z$ also lies on the surface and generates it by rotation. Now, for general a, b, c, the level set of the surface at $z = z_0$ is the ellipse

(15.42)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z_0^2}{c^2}$$

Thus, the surface is a stack of similar ellipses of size increasing as we move away from the *xy*-plane, with, again, a hyperbolic profile along each plane through the *z*-axis. We observe that this surface has the same shape as that in the case $a = b = c = 1$, except for dilations along the coordinate axes.

   Again, this surface is generated by lines as in the first case, for figure 15.22 still describes the surface, but for a change in scale in the coordinate directions. In particular, the pair of lines lying on the surface that go through the point $(a, 0, 0)$ are the lines

(15.43)
$$x = a, \qquad \frac{y}{b} = \pm\frac{z}{c}$$

Hyperboloid of one sheet $\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad$ (Figure 15.23)
The level sets $z = z_0$ of this surface are the hyperbolas

(15.44)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{z_0^2}{c^2}$$

whose vertices lie on the *x*-axis, but move further and further from the origin as z_0 moves away from the origin. To get a better view of this surface, we look at the level curves $x = x_0$:

(15.45)
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{x_0^2}{a^2} - 1$$

There is no curve for $|x| < a$, and for larger values of $|x|$, we get a family of ever increasing ellipses. This leads easily to figure 15.18.

Elliptical Cone $\qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad$ (Figure 15.24)
Here the level curves are the ever-widening ellipses

(15.46)
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

and the profile in the plane $x = 0$ is given by

(15.47)
$$\frac{y^2}{b^2} = \frac{z^2}{c^2} \quad \text{or} \quad \frac{y}{b} = \pm\frac{z}{c} ,$$

a pair of lines. The profile in the plane $y = 0$ is the pair of lines

(15.48)
$$\frac{x}{a} = \pm\frac{z}{c} .$$

This gives us figure 15.24. If $a = b$, the level sets are circles, and the profiles are the same, for the surface is the surface of revolution of the curves given by (15.48).

   Now, the above list exhausts all possibilities for surfaces given by quadratic equations of the form

(15.49)
$$Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0 ,$$

with none of A, B, C, equal to zero. Each of the surfaces illustrated in the figures has the origin as center, and has a particular axis (the z-axis, except in the case of the hyperboloid of two sheets, in which case it is the x-axis). For the general equation (15.49), the center might be any point, and the axis could be one of the other coordinate axes. For, if we complete the square in each of the variables in equation (15.49) we end up with an equation of the form

$$(15.50) \qquad A(x - x_0)^2 + B(y - y_0)^2 + C(z - z_0)^2 = H$$

If $H = 0$ we have two cases: A, B, C, all of the same sign, in which case there is no surface. Otherwise we get a cone. The axis is identified by the coefficient which is of a sign different from the other two.
If $H \neq 0$, we can divide by H, leading to one of the previous cases. The number of negative signs determines the surface; no negatives: ellipsoid; one negative: hyperboloid of one sheet; two negative: hyperboloid of two sheets.

Finally, there are two more surfaces; corresonding to the cases where one or more of the coefficients of the quadratic term is zero

Elliptical Paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z$ (Figure 15.25)
We look at the level curves $z = z_0$. If $z_0 < 0$ we get no curve. For $z_0 > 0$ we get a family of ellipses of ever increasing size. The profile on the plane $x = 0$ is the parabola $y^2 = b^2 z$, and on the $y = 0$ the parabola $x^2 = a^2 z$. This gives us enough information for figure 15.25.

Hyperbolic Paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = z$ (Figure 15.26)
We look at the level curves $z = z_0$. We get the hyperbolas

$$(15.51) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = z_0 \ .$$

If $z_0 > 0$, the axis of the hyperbola is the x axis, but if $z_0 < 0$, the axis is the y-axis. As $|z_0|$ increases, the vertices move away from the z-axis. The level curve for $z_0 = 0$ consists of the two lines $x/a = \pm y/b$. To get a better grip on the surface, we look at its profiles. The profile for $y = 0$ is the parabola $z = x^2/a^2$, which has the z-axis as axis, and opens upwards. The profile for $x = 0$ is the parabola $z = -y^2/b^2$, which opens downward. putting this information together gives us figure 15.25.

We can get a different (and perhaps more readable) view of this surface by interchanging the x and y coordinates. Figure 15.26 is that of the hyperbolic paraboloid

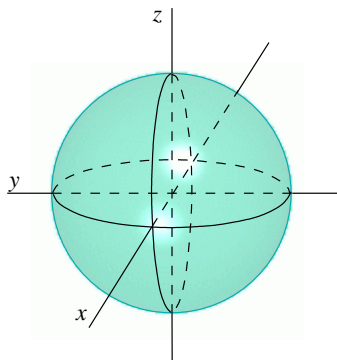$$(15.52) \qquad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = z \ .$$
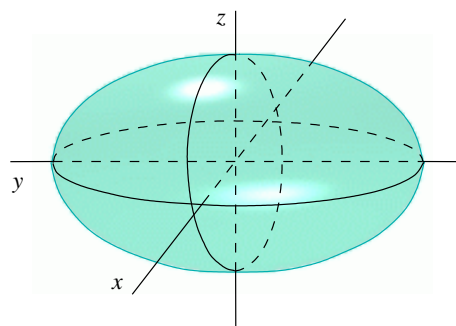
Figure 15.19: Sphere.

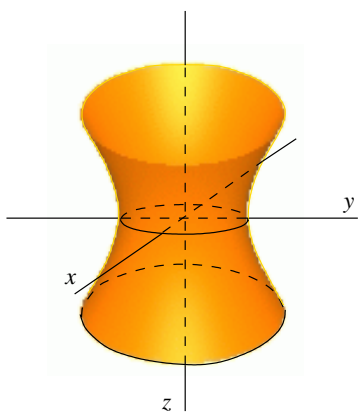Figure 15.20: Ellipsoid

Figure 15.21: Hyperboloid of one sheet

Figure 15.22: Hyperboloid of one sheet.

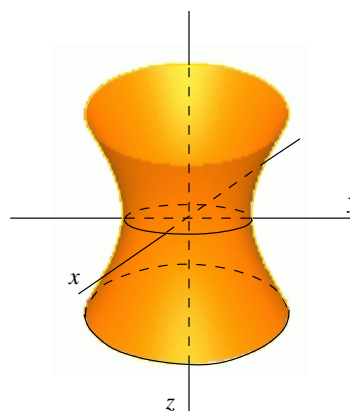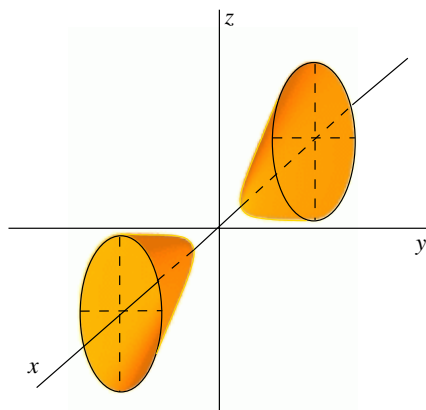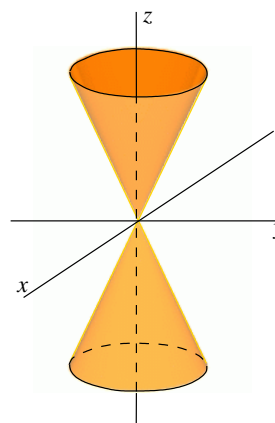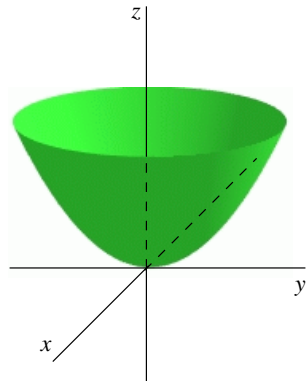Figure 15.23: Hyperboloid of two sheets

Figure 15.24: Elliptic cone

Figure 15.25: Elliptic paraboloid   Figure 15.26: Hyperboloid paraboloid