CHAPTER 14

Particles in Motion; Kepler's Laws

§14.1. Vector Functions

Vector notation is well suited to the representation of the motion of a particle. Fix a coordinate system with center \mathbf{O} , and let the position of the object at time *t*, relative to \mathbf{O} be represented by the vector function of *t*:

(14.1)
$$\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}.$$

The velocity of the object is the time rate of change of position:

(14.2)
$$\mathbf{V} = \frac{d\mathbf{X}}{dt} = \frac{dx}{dt}\mathbf{I} + \frac{dy}{dt}\mathbf{J} + \frac{dz}{dt}\mathbf{K}$$

and the **acceleration** is the time rate of change of velocity:

(14.3)
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2x}{dt^2}\mathbf{I} + \frac{d^2y}{dt^2}\mathbf{J} + \frac{d^2z}{dt^2}\mathbf{K}.$$

As shown, the differentiations are defined as they are for scalar functions, and are computed componentwise. Recall that we use the variable s to represent the length of the curve, as measured from a particular starting point. If we think of (14.1) as describing the curve parametrically, we find arc length by integrating

(14.4)
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

with respect to t. The time rate of change of distance along the curve is the **speed** of the particle. From (14.4) we see that the speed of the particle is the magnitude of the velocity vector:

(14.5)
$$\frac{ds}{dt} = |\mathbf{V}| \; .$$

Example 14.1 A particle is rotating about the circle of radius *R* at constant angular velocity ω . To find the equations of motion, we start the clock with the particle on the positive *x*-axis; that is $\mathbf{X}(0) = R\mathbf{I}$. At

time t the particle has moved through the angle ωt , so is at

(14.6)
$$\mathbf{X}(t) = R\cos\omega t\mathbf{I} + R\sin\omega t\mathbf{J}.$$

Differentiating, we find the velocity and acceleration:

(14.7)
$$\mathbf{V}(t) = -R\omega\sin\omega t\mathbf{I} + R\omega\cos\omega t\mathbf{J}, \quad \mathbf{A}(t) = -R\omega^2\cos\omega t\mathbf{I} - R\omega^2\sin\omega t\mathbf{J}.$$

Notice that $\mathbf{V}(t) = \omega \mathbf{X}(t)^{\perp}$, so $\mathbf{V}(t)$ is tangent to the circle at $\mathbf{X}(t)$ and has magnitude $\omega |\mathbf{X}(t)| = R\omega$, so the speed of the particle is $R\omega$. Also $\mathbf{A}(t) = -\omega^2 \mathbf{X}(t)$, so the acceleration is directed toward the center of the circle and has magnitude $R\omega^2$.

Example 14.2 A missile is fired on the surface of the earth at an angle of elevation α and initial speed *S* ft/sec. Find the equations of motion.



The motion is shown in figure 14.1. We take the origin of coordinates to be the initial point of the trajectory of the missile. We know that the acceleration is that due to gravity, pointing downward and of magnitude 32 ft/sec². Thus $\mathbf{A}(t) = -32\mathbf{J}$. Integrating (componentwise), we have $\mathbf{V}(t) = -32t\mathbf{J} + \mathbf{C}$ where **C** is the constant (vector) of integration. Evaluating at t = 0, we have $\mathbf{C} = \mathbf{V}(0)$, which has magnitude *S* and direction α above the horizontal. Thus

(14.8)
$$\mathbf{V}(t) = -32t\mathbf{J} + S(\cos\alpha\mathbf{I} + \sin\alpha\mathbf{J}) = S\cos\alpha\mathbf{I} + (S\sin\alpha - 32t)\mathbf{J}.$$

Integrating again

(14.9)
$$\mathbf{X}(t) = (S\cos\alpha)t\mathbf{I} + ((S\sin\alpha)t - 16t^2)\mathbf{J},$$

since $\mathbf{X}(0) = 0$. From this we can determine when the missile hits the ground again, and how far it has travelled. For $\mathbf{X}(t)$ is horizontal when its **J** component is zero; that is when $t = S \sin \alpha / 16$. The distance travelled is the **I** component of $\mathbf{X}(t)$ for this *t*, so is

(14.10)
$$d = (S\cos\alpha)\frac{S\sin\alpha}{16} = \frac{S^2\sin2\alpha}{32}$$

To choose the angle so that the horizontal distance has a maximum, we choose α so that this expression is largest. The maximum is $S^2/32$ when $\alpha = 45^\circ$.

Example 14.3 Let $\mathbf{X}(t) = \cos t \mathbf{I} + \sin t \mathbf{J} + t \mathbf{K}$. This particle is spiralling up the horizontal cylinder of radius 1 (see figure 14.2) at constant angular velocity. We have

(14.11)
$$\mathbf{V}(t) = -\sin t \mathbf{I} + \cos t \mathbf{J} + \mathbf{K}, \quad \mathbf{A}(t) = -\cos t \mathbf{I} - \sin t \mathbf{J}.$$

The speed of this particle is $|\mathbf{V}| = \sqrt{2}$, and its acceleration is of magnitude 1 and always pointed toward the *z*-axis. This is the same as circular motion in the *xy*-plane, except that the initial velocity has an elevation of 45° off the *xy*-plane.



In order to proceed we need to make some observations about the differentiation of vector functions.

Proposition 14.1 Suppose that $\mathbf{V}(t)$, $\mathbf{W}(t)$ are differentiable vector-valued functions of the variable t. *Then*

a)
$$\frac{d}{dt}(\mathbf{V} + \mathbf{W}) = \frac{d\mathbf{V}}{dt} + \frac{d\mathbf{W}}{dt}$$

If $w = w(t)$ is a differentiable scalar-valued function,
b) $\frac{d}{dt}(w\mathbf{V}) = \frac{dw}{dt}\mathbf{V} + w\frac{d\mathbf{V}}{dt}$
c) $\frac{d}{dt}(\mathbf{V} \cdot \mathbf{W}) = \frac{d\mathbf{V}}{dt} \cdot \mathbf{W} + \mathbf{V} \cdot \frac{d\mathbf{W}}{dt}$
d) $\frac{d}{dt}(\mathbf{V} \times \mathbf{W}) = \frac{d\mathbf{V}}{dt} \times \mathbf{W} + \mathbf{V} \times \frac{d\mathbf{W}}{dt}$

e) If $\mathbf{U}(t)$ is a vector of length one for all t, then \mathbf{U} and $d\mathbf{U}/dt$ are orthogonal.

The first four identities are all verified by writing all vectors in component form, and using the usual product rule for differentiation of scalar functions. e) follows from c) this way: since $\mathbf{U} \cdot \mathbf{U} = 1$, we have

(14.12)
$$\frac{d\mathbf{U}}{dt} \cdot \mathbf{U} + \mathbf{U} \cdot \frac{d\mathbf{U}}{dt} = 0$$

so that $(d\mathbf{U}/dt) \cdot \mathbf{U} = 0$.

Example 14.4 Let $\mathbf{L} = \mathbf{L}(t)$ be a unit-vector valued function; that is $|\mathbf{L}| = 1$ always. Let $\theta(t)$ be the angle between $\mathbf{L}(t)$ and the horizontal. Then

(14.13)
$$\frac{d\mathbf{L}}{dt} = \frac{d\theta}{dt}\mathbf{L}^{\perp} ,$$

where \mathbf{L}^{\perp} is the unit vector orthogonal and to the left of \mathbf{L} .

Writing **L** in components, we have

(14.14)
$$\mathbf{L}(t) = \cos \theta(t) \mathbf{I} + \sin \theta(t) \mathbf{J}, \quad \mathbf{L}^{\perp}(t) = -\sin \theta(t) \mathbf{I} + \cos \theta(t) \mathbf{J}$$

Then

(14.15)
$$\frac{d\mathbf{L}}{dt} = -\sin\theta \frac{d\theta}{dt}\mathbf{I} + \cos\theta \frac{d\theta}{dt}\mathbf{J} = \frac{d\theta}{dt}\mathbf{L}^{\perp}.$$

Example 14.5 Let $\mathbf{X}(t)$ be a vector-valued function whose values always lie in the *xy*-plane. Let A(t) be the area "swept out" by \mathbf{X} from t_0 to t; that is, the area of the domain bounded by the vectors $\mathbf{X}(t_0)$, $\mathbf{X}(t)$ and the trajectory of the vector (see figure 14.3). Show that



If we argue by differentials, this is easy. In the diagram, dA is half the area of the parallelogram spanned by **X** and $d\mathbf{X}$, so $dA = (1/2) |d\mathbf{X} \times \mathbf{X}|$.

We can also argue using the polar representation of vectors. Let r be the length of **X** and **L** the unit vector in the direction of **X**, so that $\mathbf{X} = r\mathbf{L}$. We know (the polar form for area):

(14.17)
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \,.$$

Now, using example 4,

(14.18)
$$\frac{d\mathbf{X}}{dt} = \frac{d}{dt}(r\mathbf{L}) = \frac{dr}{dt}\mathbf{L} + r\frac{d\mathbf{L}}{dt} = \frac{dr}{dt}\mathbf{L} + r\frac{d\theta}{dt}\mathbf{L}^{\perp},$$

so that

(14.19)
$$\left|\frac{d\mathbf{X}}{dt} \times \mathbf{X}\right| = \left|\left(\frac{dr}{dt}\mathbf{L} + r\frac{d\theta}{dt}\mathbf{L}^{\perp}\right) \times r\mathbf{L}\right| = r^{2}\frac{d\theta}{dt},$$

since **X** and **L** are colinear, and **L** and \mathbf{L}^{\perp} are orthogonal unit vectors.

§14.2. Planar Particle Motion

As the above examples demonstrate, a good understanding of the motion is achieved when it is described in terms of the position of the particle, rather than relative to a fixed origin. For this reason we want to represent the velocity and acceleration in terms which relate directly to the particle motion. We start with motion in the plane, with $\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J}$ representing the position of the particle relative to a given coordinate system. As above, we have the velocity and acceleration of the particle given by

(14.20)
$$\mathbf{V} = \frac{d\mathbf{X}}{dt} = \frac{dx}{dt}\mathbf{I} + \frac{dy}{dt}\mathbf{J} , \quad \mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2x}{dt^2}\mathbf{I} + \frac{d^2y}{dt^2}\mathbf{J} .$$

The speed of the moving point is

(14.21)
$$\frac{ds}{dt} = |\mathbf{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The unit vector in the direction of motion, called the tangent, is denoted T. Thus

(14.22)
$$\mathbf{V} = \frac{ds}{dt}\mathbf{T} \; .$$

Now the acceleration vector represents two aspects of the motion, describing both the way the direction of motion is turning, and the way the speed is changing. Differentiating (14.22), we have

(14.23)
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}$$

The first term gives the change of speed, and the second, the change in direction. By example 14.4, the second term is orthogonal to the first. We say that it is **normal** to the curve of motion (called the **trajectory**). Now, by example 14.4,

(14.24)
$$\frac{d\mathbf{T}}{ds} = \frac{d\theta}{ds}\mathbf{T}^{\perp}$$

where θ is the angle between **T** and the horizontal. Since both **T** and arc length are independent of time, this equation has to do only with the trajectory. We define the **curvature**, κ , of the curve as the magnitude of $d\mathbf{T}/ds$, and the **unit normal**, **N**, to the curve as the direction of $d\mathbf{T}/ds$. Thus

(14.25)
$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \,,$$

so that $\kappa = |d\mathbf{T}/ds|$ and $\mathbf{N} = \pm \mathbf{T}^{\perp}$, and equation 14.23 becomes

(14.26)
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa \mathbf{N} ,$$

since $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$, by the chain rule. The component of **A** in the direction of **T** is called the **tangential** acceleration and denoted a_T , and the component in the normal direction is the normal acceleration denoted a_N . Thus (14.26) can be rewritten as the set of equations

(14.27)
$$\mathbf{A} = a_T \mathbf{T} + a_N \mathbf{N}$$
, where $a_T = \mathbf{A} \cdot \mathbf{T} = \frac{d^2 s}{dt^2}$, $a_N = \mathbf{A} \cdot \mathbf{N} = \kappa (\frac{ds}{dt})^2$.

Example 14.6 In example 1 we considered a particle moving around a circle of radius R at constant angular velocity ω and found 2

(14.28)
$$\mathbf{V}(t) = -R\omega\sin\omega t\mathbf{I} + R\omega\cos\omega t\mathbf{J} = R\omega(-\sin\omega t\mathbf{I} + \cos\omega t\mathbf{J}),$$

(14.29)
$$\mathbf{A}(t) = -R\omega^2 \cos \omega t \mathbf{I} - R\omega^2 \sin \omega t \mathbf{J} = R\omega^2 (-\cos \omega t \mathbf{I} - \sin \omega t \mathbf{J}) .$$

Thus

(14.30)
$$\frac{ds}{dt} = |\mathbf{V}| = R\omega$$
, $\mathbf{T} = \frac{\mathbf{V}}{|\mathbf{V}|} = -\sin\omega t\mathbf{I} + \cos\omega t\mathbf{J}$, $\mathbf{N} = \mathbf{T}^{\perp} = -\cos\omega t\mathbf{I} - \sin\omega t\mathbf{J}$,

so that $\mathbf{A} = R\omega^2 \mathbf{N}$ and, by (14.65):

(14.31)
$$a_T = \frac{d^s}{dt^2} = 0, \quad a_N = \left(\frac{ds}{dt}\right)^2 \frac{1}{R} = R\omega^2, \quad \kappa = \frac{1}{R}.$$

Example 14.7 In example 14.2, we considered the trajectory of a missile fired at an initial speed of S ft/sec, and at an angle α to the horizontal. We found

(14.32)
$$\mathbf{V}(t) = S\cos\alpha\mathbf{I} + (S\sin\alpha - 32t)\mathbf{J}, \quad \mathbf{A} = -32\mathbf{J}.$$

Thus

(14.33)
$$\frac{ds}{dt} = \sqrt{(S\cos\alpha)^2 + (S\sin\alpha - 32t)^2} = S\sqrt{\cos^2\alpha + (\sin\alpha - 32t/S)^2},$$

(14.34)
$$\mathbf{T} = \frac{\cos \alpha \mathbf{I} + (\sin \alpha - 32t/S)\mathbf{J}}{\sqrt{\cos^2 \alpha + (\sin \alpha - 32t/S)(\sin \alpha - 32t/S)^2}}.$$

Now, since the acceleration is clockwise to T, so is N, thus

(14.35)
$$\mathbf{N} = -\mathbf{T}^{\perp} = \frac{(\sin\alpha - 32t/S)\mathbf{I} - \cos\alpha\mathbf{J}}{\sqrt{\cos^2\alpha + (\sin\alpha - 32t/S(\sin\alpha - 32t/S)^2}} \ .$$

Finally,

(14.36)
$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{-32(\sin \alpha - 32t/S)}{\sqrt{\cos^2 \alpha + (\sin \alpha - 32t/S(\sin \alpha - 32t/S)^2}} ,$$

(14.37)
$$a_N = \mathbf{A} \cdot \mathbf{N} = \frac{32 \cos \alpha}{\sqrt{\cos^2 \alpha + (\sin \alpha - 32t/S(\sin \alpha - 32t/S)^2}} ,$$

and the curvature is

(14.38)
$$\kappa = \frac{a_N}{(ds/dt)^2} = \frac{32\cos\alpha}{(\cos^2\alpha + (\sin\alpha - 32t/S(\sin\alpha - 32t/S)^2)^{3/2}}.$$

An effective procedure to follow to make these calculations is this:

1. Given the formula $\mathbf{X}(t)$ of motion, first differentiate it twice to obtain V and A.

2. The magnitude of V is the speed, its direction vector is T. Calculate $a_T = \mathbf{A} \cdot \mathbf{T}$.

3. The normal is $\pm \mathbf{T}^{\perp}$. Calculate $\mathbf{A} \cdot \mathbf{T}^{\perp}$; if it is positive it is a_N , and $\mathbf{N} = \mathbf{T}^{\perp}$; otherwise change the signs.

4. Finally, the curvature is $\kappa = a_N/(ds/dt)^2$.

Example 14.8 A particle moves in the plane according to the equation

(14.39)
$$\mathbf{R}(t) = t^{-1}\mathbf{I} + \ln t\mathbf{J}$$

Find the tangential and normal accelerations and the curvature of the trajectory at the time t = 1.

First we calculate the velocity and acceleration:

(14.40)
$$\mathbf{V} = \frac{-1}{t^2}\mathbf{I} + \frac{1}{t}\mathbf{J} , \quad \mathbf{A} = \frac{2}{t^3}\mathbf{I} - \frac{1}{t^2}\mathbf{J}$$

Evaluating at t = 1, we have $\mathbf{V} = -\mathbf{I} + \mathbf{J}$, $\mathbf{A} = 2\mathbf{I} - \mathbf{J}$. Then

(14.41)
$$\frac{ds}{dt} = |\mathbf{V}| = \sqrt{2} , \text{ and } \mathbf{T} = \frac{-\mathbf{I} + \mathbf{J}}{\sqrt{2}} .$$

Since **A** is clockwise to **T**, we must take

(14.42)
$$\mathbf{N} = \frac{\mathbf{I} + \mathbf{J}}{\sqrt{2}}$$

and finally

(14.43)
$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{-3}{\sqrt{2}}, \quad a_N = \mathbf{A} \cdot \mathbf{N} = \frac{1}{\sqrt{2}}, \quad \kappa = \frac{a_N}{(\frac{ds}{dt})^2} = \frac{1}{2\sqrt{2}}.$$

§14.3. Particle Motion in Space

For motion in space, the ideas are precisely the same; however, a little more work is needed to find the normal direction. Again we start with the motion described by a vector function $\mathbf{X}(t) = x(t)\mathbf{I} + y(t)\mathbf{J} + z(t)\mathbf{K}$ in a given coordinate system. The **speed** of the moving point is

(14.44)
$$\frac{ds}{dt} = |\mathbf{V}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \ .$$

The unit vector in the direction of motion (called the tangent) is denoted T. Thus

(14.45)
$$\mathbf{V} = \frac{ds}{dt}\mathbf{T} \; .$$

Since **T** is a unit vector, its derivative is orthogonal to it. We define the **normal**, **N** as the unit vector in the direction of the derivative of **T**. The plane determined by the tangent and normal vectors is called the **tangent plane** of the motion. We introduce the **curvature** of the trajectory of the motion as the magnitude of the derivative of **T** with respect to arc length; thus

(14.46)
$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \,.$$

Note that, by the chain rule, the derivative of **T** with respect to time is

(14.47)
$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \kappa \frac{ds}{dt}\mathbf{N}$$

Now the acceleration vector lies in the plane of \mathbf{T} , \mathbf{N} , which we see by differentiating (14.45):

(14.48)
$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2\kappa\mathbf{N},$$

using (14.21). The component of **A** in the direction of **T** is called the **tangential acceleration** and denoted a_T , and the component in the normal direction is the **normal acceleration** denoted a_N . Thus (14.48) can be rewritten as the set of equations

(14.49)
$$\mathbf{A} = a_T \mathbf{T} + a_N \mathbf{N}$$
 where $a_T = \mathbf{A} \cdot \mathbf{T} = \frac{d^2 s}{dt^2}$, $a_N = \mathbf{A} \cdot \mathbf{N} = \kappa (\frac{ds}{dt})^2$.

Notice that, since the normal acceleration is always positive, \mathbf{N} lies on the same side of \mathbf{T} as the acceleration \mathbf{A} . In order to find the components of the acceleration in any particular example, it is best to use

the geometry as a guide, rather than to do the calculations by direct applications of these formulae. The procedure to follow to make these calculations is this:

- 1. Given the formula $\mathbf{X}(t)$ of motion, first differentiate it twice to obtain V and A.
- 2. The magnitude of V is the speed, its direction vector is T.
- 3. Calculate $a_T = \mathbf{A} \cdot \mathbf{T}$.
- 4. From (14.49) we have

(14.50)
$$a_N \mathbf{N} = \mathbf{A} - (\mathbf{A} \cdot \mathbf{T}) \mathbf{T}$$

so calculate the vector on the right. Its magnitude is a_N and its direction vector is **N**.

5. Finally, the curvature is $\kappa = a_N/(ds/dt)^2$. Formulas for the normal acceleration and curvature which

are sometimes useful are

(14.51)
$$a_N = \frac{|\mathbf{V} \times \mathbf{A}|}{|\mathbf{V}|}, \quad \kappa = \frac{|\mathbf{V} \times \mathbf{A}|}{|\mathbf{V}|^3}.$$

To see this, start with (14.49): $\mathbf{A} = a_T \mathbf{T} + a_N \mathbf{N}$, and take the vector product with **V**. Since $\mathbf{V} \times \mathbf{T} = \mathbf{0}$,

(14.52)
$$\mathbf{V} \times \mathbf{A} = a_N \mathbf{V} \times \mathbf{N}$$

Now, take lengths: $|\mathbf{V} \times \mathbf{A}| = a_N |\mathbf{V}|$, since **V** is orthogonal to **N**, and **N** has length one. One final remark:

if the problem is to find the components of the acceleration at a particular value of t, then, immediately after differentiating, evaluate at t. Then the rest of the problem is just vector algebra with specific vectors.

Example 14.9 In example 14.3 we considered a particle moving counterclockwise up a spiral at constant speed. There we found

(14.53)
$$\mathbf{V} = \cos t \mathbf{I} - \sin t \mathbf{J} + \mathbf{K} , \quad \mathbf{A} = -\sin t \mathbf{I} - \cos t \mathbf{J} , \quad \frac{ds}{dt} = |\mathbf{V}| = \sqrt{2}$$

Note that $\mathbf{A} \cdot \mathbf{T} = 0$, $|\mathbf{A}| = 1$. Thus, since \mathbf{A} is a unit vector orthogonal to \mathbf{T} on the same side of \mathbf{T} as \mathbf{A} , it *is* the normal vector: $\mathbf{A} = \mathbf{N}$. Finally,

(14.54)
$$a_T = 0, \quad a_N = 1, \quad \kappa = \frac{a_N}{(ds/dt)^2} = \frac{1}{2}.$$

Example 14.10 A particle moves in space according to the formula

(14.55)
$$\mathbf{R}(t) = t^2 \mathbf{I} + \ln t \mathbf{J} + \frac{1}{t} \mathbf{K}$$

Find **V**, **A**, **T**, **N**, a_T , a_N , κ at the point t = 1.

First we differentiate to find the velocity and acceleration.

(14.56)
$$\mathbf{V} = 2t\mathbf{I} + \frac{1}{t}\mathbf{J} - \frac{1}{t^2}\mathbf{K} , \quad \mathbf{A} = 2\mathbf{I} - \frac{1}{t^2}\mathbf{J} + \frac{2}{t^3}\mathbf{K}$$

At t = 1, $\mathbf{V} = \mathbf{I} + \mathbf{J} - \mathbf{K}$, $\mathbf{A} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K}$, and $ds/dt = |\mathbf{V}| = \sqrt{6}$. Thus

(14.57)
$$\mathbf{T} = \frac{2\mathbf{I} + \mathbf{J} - \mathbf{K}}{\sqrt{6}} ,$$

(14.58)
$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{1}{\sqrt{6}}$$

For the normal acceleration, we calculate $a_N N$ in terms of what we already know:

(14.59)
$$a_N \mathbf{N} = \mathbf{A} - a_T \mathbf{T} = 2\mathbf{I} - \mathbf{J} + 2\mathbf{K} - \frac{1}{\sqrt{6}} \left(\frac{2\mathbf{I} + \mathbf{J} - \mathbf{K}}{\sqrt{6}}\right) = \frac{10\mathbf{I} - 7\mathbf{J} + 13\mathbf{K}}{6}$$

 a_N is the magnitude of this vector, and **N** its direction, so

(14.60)
$$a_N = \frac{\sqrt{318}}{6}, \quad \mathbf{N} = \frac{10\mathbf{I} - 7\mathbf{J} + 13\mathbf{K}}{\sqrt{318}}, \quad \kappa = \frac{a_N}{(\frac{ds}{dt})^2} = \frac{\sqrt{318}}{6}.$$

§14.4. Derivation of Kepler's Laws of Planetary Motion from Newton's Laws

§14.4.1 Historical Background

Newton's *Principia Mathematica*, published in 1684, is the fundamental text for rational mechanics, especially the dynamics of bodies in motion. It forms a bridge between late medieval rational philosophy and modern physics; written as it is in the style of its day, but in conception and execution as modern as any present day text on the subject.

In Book I of the *Principia* Newton develops the fundamentals of the theory of motion based on his methods of the Calculus and three laws of motion:

Law 1. Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Law 2. The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Law 3. To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

In Book II Newton shows how the laws of Kepler on planetary motion and Galileo's laws of falling bodies lead, in the context of the above laws, to the concept of a universal force of attraction between massive bodies. In particular, he shows that Kepler's laws imply that this force is inversely proportional to the square of the distance between the bodies. This is the hypothesis to which he is headed, his law of universal gravitation.

§14.4.2 Newton's Law of Universal Gravitation

Given two massive bodies, they exert a force on one another which is proportional to their masses and inversely proportional to the square of the distance between them. In those days this was called an "occult" force: action at a distance. Descartes' physics, based on strict ratonalism, denied the existence of mysterious, or occult causes, and thus with it, forces that act at a distance. This led Descartes to a description of planetary motion based on collisions within hypothetical vortices. In order for Newton to establish the validity of his theory, it was necessary to repudiate these Cartesian constructs, show that the concept of gravitation is forced by the work of Galileo and Kepler, and conversely, to deduce their Laws from his. These tasks he completely accomplished in Book II. It is worth noting that Newton remarks at length that, although there was no (concrete) physical evidence for gravitation, the mathematical arguments were forcing: the postulate of Universal Gravitation was forced by, and in turn predicted correctly, the accumulated data. Finally in Book III Newton showed how his Laws lead to accurate predictions of the motions of planets, and he gave rational explanations for the tides, recession and many other astronomical phenomena.

§14.4.3 Kepler's Laws

Kepler's laws are descriptive of the motion of planets around the Sun. For this discussion we shall locate the Sun at the point \mathbf{O} , and let $\mathbf{X}(t)$ represent the position of the planet (relative to \mathbf{O}) at time *t* after the

initial observation (t = 0).

Kepler I. a) A planet revolves around the Sun in an elliptical orbit.

b) The Sun is located at a focus of the ellipse.

Kepler II. The line joining the Sun to a planet sweeps out equal areas in equal times. More expicitly, let A(t) be the area swept out by the vector $\mathbf{X}(t)$. Then for any t, s, A(t+s) - A(t) = hs, where h is constant. Equivalently, dA/dt = h is constant.

Kepler III. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Newton's second law of motion is

$$\mathbf{F} = m\mathbf{A}$$

where \mathbf{F} is the force applied to the object, *m* is its mass, and \mathbf{A} is the acceleration of the object. In particular, for planetary motion, this and Newton's law of gravitation imply that the acceleration of the planet is directed toward the sun; that is, it is **centripetal**. Newton's first observation is that this is equivalent to Kepler's second law.

Proposition 14.2 Suppose the only external force exerted on an object is centripetal. Then its motion is planar and Kepler II holds.

Let X be the position vector to the object, V its velocity, and A its acceleration. Since A is colinear with X, $X \times A = 0$. Now

(14.62)
$$\frac{d}{dt}(\mathbf{X} \times \mathbf{V}) = \frac{d\mathbf{X}}{dt} \times \mathbf{V} + \mathbf{X} \times \frac{d\mathbf{V}}{dt} = \mathbf{V} \times \mathbf{V} + \mathbf{X} \times \mathbf{A} = \mathbf{0}.$$

Thus $\mathbf{X} \times \mathbf{V}$ is a constant vector \mathbf{H} . If $\mathbf{H} = \mathbf{0}$, then \mathbf{X} and \mathbf{V} are colinear, and the motion is on a line directed toward \mathbf{O} . Otherwise \mathbf{X} and \mathbf{V} always lie on the plane perpendicular to the constant vector \mathbf{H} , so the motion is planar.

Finally, let $h = |\mathbf{H}|$; h is constant, and we have

(14.63)
$$h = |\mathbf{X} \times \mathbf{V}| = \left|\mathbf{X} \times \frac{d\mathbf{X}}{dt}\right| = r^2 \frac{d\theta}{dt} = 2\frac{dA}{dt}$$

(by example 14.5), giving us Kepler's second law. We now show that the assumption of planar motion

and Kepler's second law implies the force is centripetal. The assumption that the motion is planar implies that $\mathbf{X} \times \mathbf{V}$ has a fixed direction, and Kepler's second law implies (see example 14.5) that the length of that vector is constant. Thus $\mathbf{X} \times \mathbf{V}$ is a constant vector. Differentiating, we obtain

$$\mathbf{V} \times \mathbf{V} + \mathbf{X} \times \mathbf{A} = \mathbf{0},$$

so $X \times A = 0$, and A and X are colinear, that is, A is centripetal. As an aside, we observe that the fact that $X \times V$ is constant can be interpreted as the conservation of angular momentum. The crux of

Newton's argument is that the first Kepler Law follows from the inverse-square law. We shall go straight to this argument, although Newton did take the trouble to also show that Kepler's first law forces the inverse-square law.

We continue the argument of Proposition 14.2, continuing with that notation. The additional hypothesis is that

(14.65)
$$\frac{d\mathbf{V}}{dt} = -\frac{k}{r^2}\mathbf{L},$$

where k is a positive constant. The trick now is to take the cross product with **K**. Even though the action takes place completely in the plane perpendicular to **K**, this allows us to introduce the area information in vectorial form.

(14.66)
$$\frac{d\mathbf{V}}{dt} \times \mathbf{K} = -\frac{k}{r^2} \mathbf{L} \times \mathbf{K} = \frac{k}{r^2} \mathbf{L}^{\perp} ,$$

since L and K are orthogonal unit vectors and the system L, K, L^{\perp} is left-handed. But, as we saw in example 14.4,

(14.67)
$$\frac{d\mathbf{L}}{dt} = \frac{d\theta}{dt}\mathbf{L}^{\perp} \quad \text{or} \quad \mathbf{L}^{\perp} = \frac{d\mathbf{L}}{dt} / \frac{d\theta}{dt} \; .$$

By (14.63), $r^2 d\theta / dt = h$, so

(14.68)
$$\frac{d\mathbf{V}}{dt} \times \mathbf{K} = \frac{k}{h} \frac{d\mathbf{L}}{dt},$$

Now, integrating (19), (since \mathbf{K} , k, h are constant),

(14.69)
$$\mathbf{V} \times \mathbf{K} = \frac{k}{h} (\mathbf{L} + \mathbf{E}),$$

where **E** is a constant vector in the plane of action. Take the dot product with **X**:

(14.70)
$$\mathbf{X} \cdot \mathbf{V} \times \mathbf{K} = \frac{k}{h} \mathbf{X} \cdot (\mathbf{L} + \mathbf{E})$$

Observe that

(14.71)
$$\mathbf{X} \cdot \mathbf{V} \times \mathbf{K} = (\mathbf{X} \times \mathbf{V}) \cdot \mathbf{K} = h\mathbf{K} \cdot \mathbf{K} = h,$$

since $\mathbf{X} \times \mathbf{V} = h\mathbf{K}$. This gives

(14.72)
$$\mathbf{X} \cdot (\mathbf{L} + \mathbf{E}) = \frac{h^2}{k}.$$

The motion therefore always satisfies this equation. We now show that it describes an ellipse. We move to polar coordinates, taking the origin at the Sun, and the ray $\theta = 0$ in the direction of **E**.

Figure 14.4



We have: $\mathbf{E} = e\mathbf{I}$, $\mathbf{L} = \cos\theta\mathbf{I} + \sin\theta\mathbf{J}$ and $\mathbf{X} = r\mathbf{L}$. Then $\mathbf{X} \cdot \mathbf{L} = r$, $\mathbf{X} \cdot \mathbf{E} = r\cos\theta$ and equation (14.72) becomes

(14.73)
$$r(1 + e\cos\theta) = \frac{h^2}{k}$$

which is the equation of a conic with eccentricity e, the x-axis as major axis and a focus at the origin. If e < 1 this is an ellipse, if e = 1 a parabola, and if e > 1 a hyperbola. Since the planets have recurring orbits, the only possibility is that of an ellipse, e < 1.

Finally, we can derive Kepler's third law as a consequence, either of his first and second laws, or Newton's Law of Gravitational Attraction.

Proposition 14.4 Let T be the length of the planetary year, a and b the lengths of the half-major and half-minor axes of the orbit of the planet. Then

(14.74)
$$T^2 = \frac{4\pi^2}{k}a^3 \,.$$

The important thing here is that the equation which asserts Kepler II,

(14.75)
$$\frac{dA}{dt} = \frac{1}{2}h,$$

allows us to relate the area (πab) of the ellipse with the time it takes to sweep out one full orbit, the planetary year *T*. Integrate with respect to *t* over a full cycle:

(14.76)
$$\pi ab = \frac{1}{2}hT \; .$$

The form of the equation of the orbit (14.73) allows us to replace *h* in this equation by the gravitational constant *k* relating the planet to the sun. We know that the orbit crosses the major axis at the points $\theta = 0$ and $\theta = \pi$, and the distance between these two points is the major axis. This distance (the length of the major axis) is the sum of the values of *r* at $\theta = 0$, π :

(14.77)
$$2a = \frac{h^2}{k(1+e)} + \frac{h^2}{k(1-e)}$$

so $a = h^2/k(1-e^2)$. Using the relation $b^2 = a^2(1-e^2)$, this gives us

$$\frac{b^2}{a} = \frac{h^2}{k} \ .$$

Combining (6.9) and (6.10) gives us the desired result:

(14.79)
$$T^{2} = \left(\frac{2\pi ab}{h}\right)^{2} = (2\pi ab)^{2} \frac{a}{kb^{2}} = \frac{4\pi^{2}}{k}a^{3}.$$

Note (in (14.73)) that since h/2 is the rate of change of the swept-out area, and e, the eccentricity of the ellipse, is related to the major and minor axes, these can be computed by terrestrial measurements for any planet (and in fact, had been by Kepler). Using his law of universal gravitiation and other known cosmological computations, Newton was able to estimate the constant k as well, so he was able to give good estimates for the actual orbits of the planets. These estimates agreed quite strongly with observations. Since Newton could not exclude the possibility of parabolic or hyperbolic orbits, he hypothesized that these were the orbits for some comets. Using the observations of the comet of 1681-2, (and Kepler's third law) he discovered its orbit to be elliptical and fairly accurately predicted the date of its return.