CHAPTER 13

Vector Algebra

§13.1. Basic Concepts

A *vector* $\mathbf{V}$ in the plane or in space is an arrow: it is determined by its length, denoted $|\mathbf{V}|$ and its direction. Two arrows represent the same vector if they have the same length and are parallel (see figure 13.1). We use vectors to represent entities which are described by magnitude and direction. For example, a force applied at a point is a vector: it is completely determined by the magnitude of the force and the direction in which it is applied. An object moving in space has, at any given time, a direction of motion, and a speed. This is represented by the velocity vector of the motion. More precisely, the velocity vector at a point is an arrow of length the speed (ds/dt), which lies on the tangent line to the trajectory. The success and importance of vector algebra derives from the interplay between geometric interpretation and algebraic calculation. In these notes, we will define the relevant concepts geometrically, and let this lead us to the algebraic formulation.

Figure 13.1

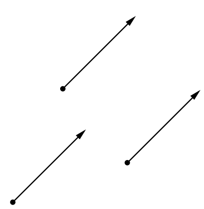
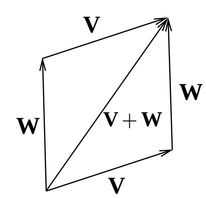


Figure 13.2



Newton did not write in terms of vectors, but through his diagrams we see that he clearly thought of forces in these terms. For example, he postulated that two forces acting simultaneously can be treated as acting sequentially. So suppose two forces, represented by vectors $\mathbf{V}$ and $\mathbf{W}$, act on an object at a particular point. What the object feels is the *resultant* of these two forces, which can be calculated by placing the vectors end to end (as in figure 13.2). Then the resultant is the vector from the initial point of the first vector to the end point of the second. Clearly, this is the same if we reverse the order of the vectors. We call this the **sum** of the two vectors, denoted $\mathbf{V}+\mathbf{W}$. For example, if an object is moving in a fluid in space with a velocity $\mathbf{V}$, while the fluid is moving with velocity $\mathbf{W}$, then the object moves (relative to a fixed point) with velocity $\mathbf{V}+\mathbf{W}$.

Definition 13.1
a) A **vector** *represents the length and direction of a line segment. The* **length** *is denoted* $|\mathbf{V}|$*. A* **unit vector** $\mathbf{U}$ *is a vector of length 1. The* **direction** *of a vctor* $\mathbf{V}$ *is the unit vector* $\mathbf{U}$ *parallel to* $\mathbf{V}$*:* $\mathbf{U} = \mathbf{V}/|\mathbf{V}|$*.*
b) Given two points P, Q, the vector from P to Q is denoted $\vec{PQ}$*.*
c) Addition. The **sum***, or* **resultant***,* $\mathbf{V}+\mathbf{W}$ *of two vectors* $\mathbf{V}$ *and* $\mathbf{W}$ *is the diagonal of the parallelogram with sides* $\mathbf{V},\mathbf{W}$*.*
d) Scalar Multiplication. To distinguish them from vectors, real numbers are called **scalars***. If c is a positve real number,* $c\mathbf{V}$ *is the vector with the same direction as* $\mathbf{V}$ *and of length* $c|\mathbf{V}|$*. If c negative, it is the same, but directed in the opposite direction.*

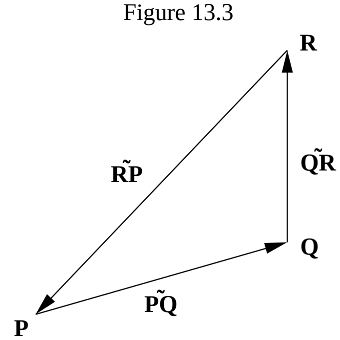
We note that the vectors $\mathbf{V}$, $c\mathbf{V}$ are parallel, and conversely, if two vectors are parallel (that is, they have the same direction), then one is a scalar multiple of the other.

Example 13.1 Let $P, \, ,Q, \, R$ be three points in the plane not lying on a line. Then

$$\vec{PQ}+\vec{QR}+\vec{RP} = \mathbf{0}\ . \tag{13.1}$$

From figure 13.3, we see that the vector $\vec{RP}$ is the same line segment as $\vec{PQ}+\vec{QR}$, but points in the opposite direction. Thus $\vec{RP} = -(\vec{PQ}+\vec{QR})$.

Figure 13.3



Example 13.2 Using vectors, show that if two triangles have corresponding sides parallel, that the lengths of corresponding sides are proportional.

Represent the sides of the two triangles by $\mathbf{U},\ \mathbf{V},\ \mathbf{W}$ and $\mathbf{U}',\ \mathbf{V}',\ \mathbf{W}'$ respectively. The hypothesis is that there are scalars $a,\ b,\ c$ such that $\mathbf{U}' = a\mathbf{U},\ \mathbf{V}' = b\mathbf{U},\ \mathbf{W}' = c\mathbf{W}$. The conclusion is that $a = b = c$. To show this, we start with the result of example 1; since these are the sides of a triangle, we have

$$\mathbf{U}+\mathbf{V}+\mathbf{W} = \mathbf{0}\ , \quad \mathbf{U}'+\mathbf{V}'+\mathbf{W}' = \mathbf{0}\ , \quad \text{or, what is the same,} \quad a\mathbf{U}+b\mathbf{V}+c\mathbf{W} = \mathbf{0} \tag{13.2}$$

The first equation gives us $\mathbf{U} = -\mathbf{V}-\mathbf{W}$, which, when substituted in the last equation gives

$$(b-a)\mathbf{V}+(c-a)\mathbf{W} = \mathbf{0} \tag{13.3}$$

Now, if $b \neq a$, this tells us that $\mathbf{V}$ and $\mathbf{W}$ are parallel, and so the triangle lies on a line: that is, there is no triangle. Thus we must have $b = a$, and by the same reasoning, $c = a$ also.

§13.2. Vectors in the Plane

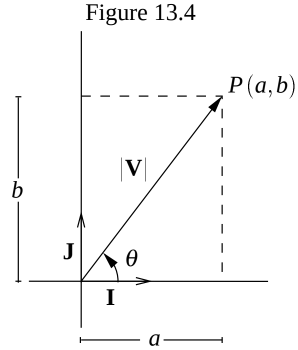
The advantage gained in using vectors is that they are moveable, and not tied to any particular coordinate system. As we have seen in the examples of the previous section, geometric facts can be easily derived using vectors while working in coordinates may be cumbersome. However, it is often the case, that in working with vectors we must do calculations in a particular coordinate system. It is important to realize that it is the worker who gets to choose the coordinates; it is not necessarily inherent in the problem.

We now explain how to move back and forth between vectors and coordinates. Suppose, then, that a coordinate system has been chosen: a point O, the origin, and two perpendicular lines through the origin, the x- and y-axes. A vector $\mathbf{V}$ is determined by its length, $|\mathbf{V}|$ and its direction, which we can describe by the angle θ that $\mathbf{V}$ makes with the horizontal (see figure 13.4). In this figure, we have realized $\mathbf{V}$ as the vector $\vec{OP}$ from the origin to P. Let (a,b) be the cartesian coordinates of P. Note that $\mathbf{V}$ can be realized as the sum of a vector of length a along the x-axis, and a vector of length b along the y-axis. We express this as follows.

Definition 13.2 *We let* $\mathbf{I}$ *represent the vector from the origin to the point (1,0), and* $\mathbf{J}$ *the vector from the origin to the point (0,1). These are the* **basic** *unit vectors (a unit vector is a vector of length 1). The unit vector in the direction* θ *is* $\cos\theta\mathbf{I}+\sin\theta\mathbf{J}$.

If $\mathbf{V}$ is a vector of length r and angle θ, then $\mathbf{V}=r(\cos\theta\mathbf{I}+\cos\theta\mathbf{J})$. If $\mathbf{V}$ is the vector from the origin to the point (a,b); r is the length of $\mathbf{V}$, and $\cos\theta\mathbf{I}+\cos\theta\mathbf{J}$ is its direction. If $P(a,b)$ is the endpoint of $\mathbf{V}$, then $\mathbf{V}=\vec{OP}=a\mathbf{I}+b\mathbf{J}$. a and b are called the **components** of $\mathbf{V}$.

Figure 13.4



Of course, r and θ are the usual polar coordinates, and we have these relations:

$$|\mathbf{V}|=\sqrt{a^2+b^2}\ ,\quad \theta=\arctan\frac{b}{a}\ ,\qquad a=|\mathbf{V}|\cos\theta\ ,\quad b=|\mathbf{V}|\sin\theta\ . \tag{13.4}$$

We add vectors by adding their components, and multiply a vector by a scalar by multiplying the components by the scalar.

Proposition 13.2 *If* $\mathbf{V}=a\mathbf{I}+b\mathbf{J}$ *and* $\mathbf{W}=c\mathbf{I}+d\mathbf{J}$*, then* $\mathbf{V}+\mathbf{W}=(a+c)\mathbf{I}+(b+d)\mathbf{J}$.

This is verified in figure 13.5.

Figure 13.5

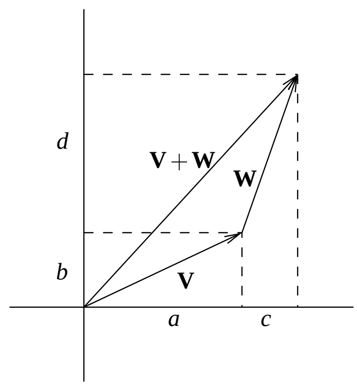
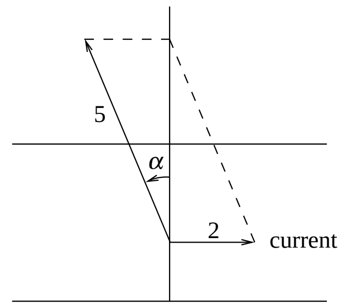


Figure 13.6



Example 13.3 A boy can paddle a canoe at 5 mph. Suppose he wants to cross a river whose current moving at 2 mph. At what angle to the perpendicular from one bank to the other should he direct his canoe?

Draw a diagram so that the river is moving horizontally from left to right, and the direct crossing is vertical (see figure 13.6). Place the origin on the lower bank of the river, and choose the x-axis in the direction of flow, and the y-axis perpendicularly across the river. TIn these coordinates, the velocity vector of the current is $2\mathbf{I}$. Let $\mathbf{V}$ be the velocity vector of the canoe. We are given that $|\mathbf{V}| = 5$ and we want the resultant of the two velocities to be vertical. If α is the desired angle, we see from the diagram that $\sin\alpha = 2/5$, so $\alpha = 23.5°$.

Example 13.4 An object on the plane is subject to the three forces $\mathbf{F} = 2\mathbf{I} + \mathbf{J}$, $\mathbf{G} = -8\mathbf{J}$, $\mathbf{H}$. Assuming the object doesn't move, find $\mathbf{H}$. At what angle to the horizontal is $\mathbf{H}$ directed?

By Newton's law, the sum of the forces must be zero. Thus

$$\mathbf{H} = -\mathbf{F} - \mathbf{G} = -2\mathbf{I} - \mathbf{J} + 8\mathbf{J} == -2\mathbf{I} + 7\mathbf{J}\ . \tag{13.5}$$

If α is the angle from the positive x-axis to $\mathbf{H}$, $\tan\alpha = -7/2$, so $\alpha = 105.95°$, since $\mathbf{H}$ points upward and to the left.

Since vectors represent magnitude and length, we need a computationally straightforward way of determining lengths and angles, given the components of a vector.

Definition 13.3 *The* **dot product** *of two vectors* $\mathbf{V_1}$ *and* $\mathbf{V_2}$ *is defined by the equation*

$$\mathbf{V_1} \cdot \mathbf{V_2} = |\mathbf{V_1}||\mathbf{V_2}|\cos\beta\ , \tag{13.6}$$

where β *is the angle between the two vectors.*

Note that since the cosine is an even function, it does not matter if we take β from $\mathbf{V_1}$ to $\mathbf{V_2}$, or in the opposite sense. In particular, we see that $\mathbf{V_1} \cdot \mathbf{V_2} = \mathbf{V_2} \cdot \mathbf{V_1}$. Now, we see how to write the dot product in terms of the components of the two vectors.

Proposition 13.3 *Let* $\mathbf{V_1} = a_1\mathbf{I} + b_1\mathbf{J}$ *and* $\mathbf{V_2} = a_2\mathbf{I} + b_2\mathbf{J}$. *Then*

$$\mathbf{V_1} \cdot \mathbf{V_2} = a_1 a_2 + b_1 b_2 \tag{13.7}$$

with equality holding only when the vectors are parallel.

To see this, we use the polar representation of the vectors:

(13.8) $$\mathbf{V_1} = r_1(\cos\theta_1\mathbf{I} + \sin\theta_1\mathbf{J}) , \quad \mathbf{V_2} = r_2(\cos\theta_2\mathbf{I} + \sin\theta_2\mathbf{J}) .$$

Then

(13.9) $$\mathbf{V_1}\cdot\mathbf{V_2} = r_1r_2\cos(\theta_1-\theta_2) = r_1r_2\cos\theta_1\cos\theta_2 + r_1r_2\sin\theta_1\sin\theta_2$$

by the addition formula for the cosine. This is the same as

(13.10) $$\mathbf{V_1}\cdot\mathbf{V_2} = (r_1\cos\theta_1)(r_2\cos\theta_2) + (r_1\sin\theta_1)(r_2\sin\theta_2)$$

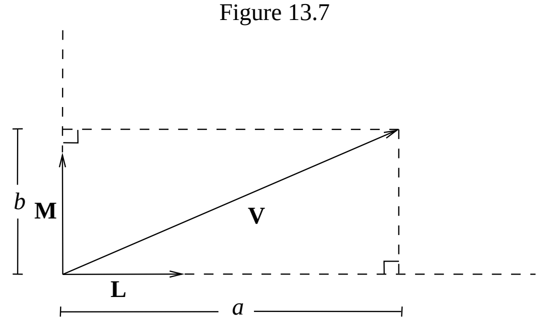
which is equation (13.7) in Cartesian coordinates. As for the last statement, we have strict inequality unless $\cos\beta = 1$, that is $\beta = 0$ or π, in which case the vectors are parallel.

Proposition 13.4
a) Two vectors $\mathbf{V}$ *and* $\mathbf{W}$ *are orthogonal if and only if* $\mathbf{V}\cdot\mathbf{W} = 0$*.*
b) If $\mathbf{L}$ *and* $\mathbf{M}$ *are two unit vectors with* $\mathbf{L}\cdot\mathbf{M} = 0$*, then for any vector* $\mathbf{V}$*, we can write*

(13.11) $$\mathbf{V} = a\mathbf{L} + b\mathbf{M} , \quad \text{with} \quad a = \mathbf{V}\cdot\mathbf{L},\ b = \mathbf{V}\cdot\mathbf{M} \text{ , and } |\mathbf{V}| = \sqrt{a^2+b^2} .$$

We shall say that a pair of unit vectors $\mathbf{L}$, $\mathbf{M}$ with $\mathbf{L}\cdot\mathbf{M} = 0$ form a **base** for the plane. This statement just reiterates that we can put cartesian coordinates on the plane with any point as origin and coordinate axes two orthogonal lines through the origin; that is the lines in the directions of $\mathbf{L}$ and $\mathbf{M}$. To show part b) we start with figure 13.7.

Figure 13.7



From that figure, we see that we can write any vector as a sum $\mathbf{V} = a\mathbf{L} + b\mathbf{M}$ with (by the Pythagorean theorem) $|\mathbf{V}| = \sqrt{a^2+b^2}$. We now show that a, b are as described;

(13.12) $$\mathbf{V}\cdot\mathbf{L} = (a\mathbf{L} + b\mathbf{M})\cdot\mathbf{L} = a\mathbf{L}\cdot\mathbf{L} + b\mathbf{M}\cdot\mathbf{L} = a .$$

Similarly $\mathbf{V}\cdot\mathbf{M} = b$.

Example 13.5 Find the angle β between the vectors $\mathbf{V} = 2\mathbf{I} - 3\mathbf{J}$ and $\mathbf{W} = \mathbf{I} + 2\mathbf{J}$.
We have $|\mathbf{V}| = \sqrt{2^2+3^2} = \sqrt{13}$, $|\mathbf{W}| = \sqrt{1^2+2^2} = \sqrt{5}$ and $\mathbf{V}\cdot\mathbf{W} = 2(1) + (-3)(2) = -4$. Thus

(13.13) $$\cos\beta = \frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{V}||\mathbf{W}|} = \frac{-4}{\sqrt{65}} = -.496$$

so $\beta = -119.7°$.

Example 13.6 Suppose we have put cartesian coordinates on the plane, with $\mathbf{I}$, $\mathbf{J}$ the standard base. Let

$$\mathbf{L} = \frac{\mathbf{I}+\mathbf{J}}{\sqrt{2}}\ , \quad \mathbf{M} = \frac{-\mathbf{I}+\mathbf{J}}{\sqrt{2}} \tag{13.14}$$

be a new base. Given the point $P(5,2)$, write $\vec{OP}$ in terms of $\mathbf{L}$ and $\mathbf{M}$.

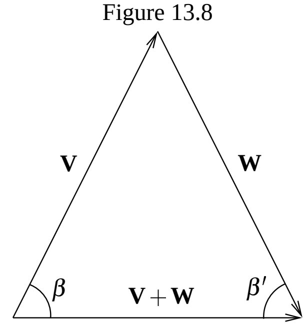
By the preceding proposition,

$$\vec{OP}\cdot\mathbf{L} = (5\mathbf{I}+2\mathbf{J})\cdot\Big(\frac{\mathbf{I}+\mathbf{J}}{\sqrt{2}}\Big) = \frac{7}{\sqrt{2}}\ , \quad \vec{OP}\cdot\mathbf{M} = (5\mathbf{I}+2\mathbf{J})\cdot\Big(\frac{-\mathbf{I}+\mathbf{J}}{\sqrt{2}}\Big) = -\frac{3}{\sqrt{2}}\ , \tag{13.15}$$

so $\vec{OP} = (7\mathbf{L}-3\mathbf{M})/\sqrt{2}$.

Example 13.7 Show, using vectors, that the interior angles of an isosceles triangle are equal.

Figure 13.8



In figure 13.8 we have labelled the sides of equal length as $\mathbf{V}$ and $\mathbf{W}$. Thus, the base of the triangle is $\mathbf{V}+\mathbf{W}$. First of all, since $|\mathbf{V}| = |\mathbf{W}|$, we have $(\mathbf{V}+\mathbf{W})\cdot\mathbf{V} = \mathbf{V}\cdot\mathbf{V}+\mathbf{W}\cdot\mathbf{V} = \mathbf{W}\cdot\mathbf{W}+\mathbf{V}\cdot\mathbf{W} = (\mathbf{V}+\mathbf{W})\cdot\mathbf{W}$. Thus, by (2),

$$\cos\beta = \frac{(\mathbf{V}+\mathbf{W})\cdot\mathbf{V}}{|\mathbf{V}+\mathbf{W}||\mathbf{V}|} = \frac{(\mathbf{V}+\mathbf{W})\cdot\mathbf{W}}{|\mathbf{V}+\mathbf{W}||\mathbf{W}|} = \cos\beta'\ . \tag{13.16}$$

Since both angles are acute, $\beta = \beta'$.

Example 13.8 Find a vector orthogonal to $\mathbf{V} = 3\mathbf{I}+4\mathbf{J}$ and of the same length.

The vectors $\mathbf{V} = a\mathbf{I}+b\mathbf{J}$, $\mathbf{W} = c\mathbf{I}+d\mathbf{J}$, are orthogonal precisely when $ac+bd = 0$. Thus, if we are given a,b, we take $c = -b$, $d = a$ to get an orthogonal vector. So for this example, we can take $\mathbf{W} = -4\mathbf{I}+3\mathbf{J}$. Clearly, since the coefficients are the same but for sign, $|\mathbf{W}| = |\mathbf{V}|$. We could also take the vector in the opposite direction: $-\mathbf{W} = 4\mathbf{I}-3\mathbf{J}$

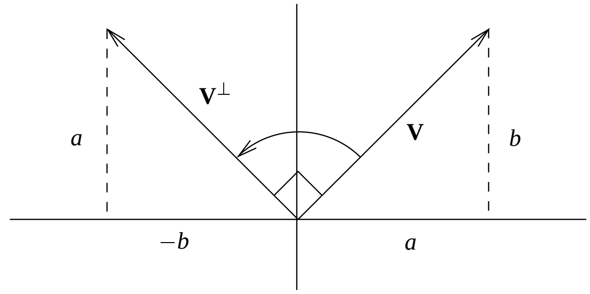
In general, if $\mathbf{V} = c\mathbf{I}+d\mathbf{J}$ then both $-d\mathbf{I}+c\mathbf{J}$ and $d\mathbf{I}-c\mathbf{J}$ are orthogonal to $\mathbf{V}$ and of the same length. The first is counterclockwise to $\mathbf{V}$, and the second, clockwise.

Definition 13.4 *Given the vector* $\mathbf{V}$*, we shall denote by* $\mathbf{V}^\perp$ *that vector which is orthogonal to, of the same length as, and counterclockwise to* $\mathbf{V}$*. In components, we have:*

$$\text{If}\quad \mathbf{V} = a\mathbf{I}+b\mathbf{J}\ , \quad\text{then}\quad \mathbf{V}^\perp = -b\mathbf{I}+a\mathbf{J} \tag{13.17}$$

See figure 13.8 to see that $\mathbf{V}^\perp$ is counterclockwise to $\mathbf{V}$ (at least in the case where both a and b are positive).

Figure 13.9

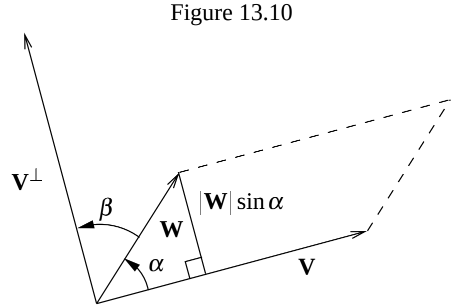


Definition 13.5 *Given two vectors* $\mathbf{V}$ *and* $\mathbf{W}$*, we define the* **determinant** $\det(\mathbf{V},\mathbf{W})$ *of the two vectors as the signed area of the parallelogram spanned by the two vectors. The sign is positive if* $\mathbf{W}$ *is counterclockwise from* $\mathbf{V}$*; otherwise negative.*

In figure 13.10, α is the angle from $\mathbf{V}$ to $\mathbf{W}$. Thus

$$\det(\mathbf{V},\mathbf{W}) = |\mathbf{V}||\mathbf{W}|\sin\alpha\ . \tag{13.18}$$

Figure 13.10



Now, let β be the angle from $\mathbf{W}$ to $\mathbf{V}^\perp$ so that (in figure 13.10), $\alpha+\beta=\pi/2$, and we have $\sin\alpha = \cos\beta$. Since $|\mathbf{V}| = |\mathbf{V}^\perp|$, we can rewrite (13.18) as

$$\det(\mathbf{V},\mathbf{W}) = |\mathbf{V}^\perp||\mathbf{W}|\cos\beta = \mathbf{V}^\perp\cdot\mathbf{W} \tag{13.19}$$

This gives us the following.

Proposition 13.5 *The determinant of the two vectors* $\mathbf{V} = a\mathbf{I}+b\mathbf{J}$ *and* $\mathbf{W} = c\mathbf{I}+d\mathbf{J}$ *is the determinant of the matrix whose rows are the vectors* $\mathbf{V}$ *and* $\mathbf{W}$*:*

$$\det(\mathbf{V},\mathbf{W}) = ad - bc \tag{13.20}$$

For, $\mathbf{V}^\perp = -b\mathbf{I}+a\mathbf{J}$*, and from (13.19),* $\det(\mathbf{V},\mathbf{W}) = \mathbf{V}\cdot\mathbf{W}^\perp = -bc+ad = ad-bc$.

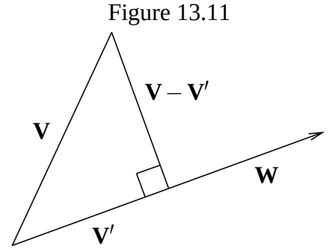
The vectors $\mathbf{V}$ and $\mathbf{W}$are parallel (or collinear) if and only if $\det(\mathbf{V},\mathbf{W}) = 0$, for in this case there is no parallelogram. We also have the inequality

$$|\det(\mathbf{V},\mathbf{W})| \leq |\mathbf{V}||\mathbf{W}| \ , \tag{13.21}$$

with equality holding if and only if $\mathbf{V}$ and $\mathbf{W}$ are orthogonal.

Definition 13.6 *Given two vectors* $\mathbf{V}$ *and* $\mathbf{W}$*, the* **projection** *of* $\mathbf{V}$ *in the direction of* $\mathbf{W}$ *is that vector* $\mathbf{V}'$ *parallel to* $\mathbf{W}$ *such that* $\mathbf{V}-\mathbf{V}'$ *is orthogonal to* V' *(see figure 13.11).*

Figure 13.11



Proposition 13.6 *The projection* $\mathbf{V}'$ *of* $\mathbf{V}$ *in the direction of* $\mathbf{W}$ *is given by the formula*

$$\mathbf{V}' = pr_{\mathbf{W}}(\mathbf{V}) = \frac{\mathbf{V}\cdot\mathbf{W}}{\mathbf{W}\cdot\mathbf{W}}\mathbf{W} \ . \tag{13.22}$$

If $\mathbf{U}$ *is a unit vector in the direction of* $\mathbf{W}$*, then*

$$\mathbf{V}' = (\mathbf{V}\cdot\mathbf{U})\mathbf{U} \ , \quad \text{and} \quad \mathbf{V} = (\mathbf{V}\cdot\mathbf{U})\mathbf{U} + (\mathbf{V}\cdot\mathbf{U}^{\perp})\mathbf{U}^{\perp} \ . \tag{13.23}$$

To show this we start with the equation $(\mathbf{V}-\mathbf{V}')\cdot\mathbf{V}' = \mathbf{0}$. Since $\mathbf{V}' = a\mathbf{W}$ for some a, this gives us

$$(\mathbf{V}-a\mathbf{W})\cdot a\mathbf{W} = 0 \ , \quad \text{or} \quad a^2\mathbf{W}\cdot\mathbf{W} = a\mathbf{V}\cdot\mathbf{W} \tag{13.24}$$

If $a = 0$, then $\mathbf{V}' = \mathbf{0}$ and $\mathbf{V}$ and $\mathbf{W}$ are orthogonal. Otherwise

$$a = \frac{\mathbf{V}\cdot\mathbf{W}}{\mathbf{W}\cdot\mathbf{W}} \ , \tag{13.25}$$

giving us (13.22). The rest of the proposition follows by replacing $\mathbf{W}$ by the unit vector $\mathbf{U}$, and should be viewed as a restatement of Proposition 13.6.

Example 13.9 Find the area of the parallelogram whose vertices are at $O(0,0), P(4,-2), \ Q(5,8), R(9,6)$.

This is the parallelogram determined by the vectors from the origin O to the points P and Q: $\vec{OP} = 4\mathbf{I}-2\mathbf{J}$, $\vec{OQ} = 5\mathbf{I}-8\mathbf{J}$, so has signed area $4(-8)-(-2)(5) = -22$. We verify these are the vertices of a parallelogram by calculating $\vec{OP}+\vec{OQ} = 9\mathbf{I}+6\mathbf{J} = \vec{OR}$.

In order to discuss geometric objects in the coordinate plane, it is useful to represent a point $X(x,y)$ by the vector $\mathbf{X} = \vec{OX} = x\mathbf{I}+y\mathbf{J}$ from the origin to X. For Y another point, the vector from X to Y is thus represented by $\mathbf{Y}-\mathbf{X}$ (see figure 13.12).

Figure 13.12

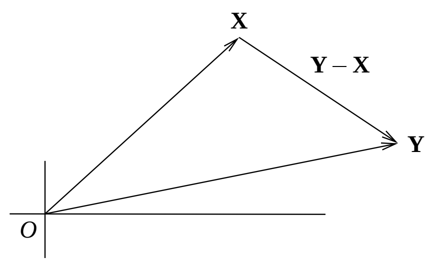
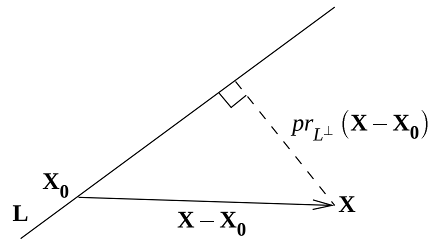


Figure 13.13



A line L is determined by its direction and a point on the line. let $\mathbf{X_0}$ be a point on L, and $\mathbf{L}$ a vector parallel to the line L. Then, for any point $\mathbf{X}$, it is on the line if and only if $\mathbf{X}-\mathbf{X_0}$ is parallel to $\mathbf{L}$, or, what is the same, orthogonal to $\mathbf{L}^\perp$. This leads to these two equations, called the equation of the line:

(13.26) $$(\mathbf{X}-\mathbf{X_0})\cdot\mathbf{L}^\perp = 0 \qquad \text{or} \qquad \det(\mathbf{X}-\mathbf{X_0},\mathbf{L}) = 0\ .$$

Also, since $\mathbf{X}-\mathbf{X_0}$ is parallel to L if and only if $\mathbf{X}-\mathbf{X_0}$ is a scalar multiple of $\mathbf{L}$, we have the *parametric form* of the equation of the line:

(13.27) $$L: \qquad \mathbf{X} = \mathbf{X_0} + t\mathbf{L}\ .$$

A line is also determined by two points $\mathbf{X_0}$, $\mathbf{X_1}$ on the line. Given that information, we find the equations of the line by taking $\mathbf{L} = \mathbf{X_1}-\mathbf{X_0}$.

Now, suppose L is a line and $\mathbf{X}$ is a point not on the line. We seek a formula for the distance from the point $\mathbf{X}$ to the line. We see from figure 13.13 that this is the length of the projection in the direction perpendicular to L of a vector from $\mathbf{X}$ to any point $\mathbf{X_0}$ on L. This leads to the formula for the distance from $\mathbf{X}$ to L

(13.28) $$d(\mathbf{X},L) = |pr_{\mathbf{L}^\perp}(\mathbf{X}-\mathbf{X_0})|$$

Example 13.10 Let L be the line given by the equation $3x-y=7$. Find the distance from (2,4) to L.

By comparison with equation (13.26) we see that $\mathbf{L}^\perp = 3\mathbf{I}-\mathbf{J}$. To use (13.28) we need a point on the line; any solution of the equation $3x-y=7$ will do. (3,2) is a solution, so we take $\mathbf{X_0} = 3\mathbf{I}+2\mathbf{J}$. Thus, for our point, $\mathbf{X} = 2\mathbf{I}+4\mathbf{J}$, the distance is

(13.29) $$|pr_{\mathbf{L}^\perp}(\mathbf{X}-\mathbf{X_0})| = \frac{|(\mathbf{X}-\mathbf{X_0})\cdot\mathbf{L}^\perp|}{|\mathbf{L}^\perp|} = \frac{|(-\mathbf{I}+2\mathbf{J})\cdot(3\mathbf{I}-\mathbf{J})|}{|3\mathbf{I}-\mathbf{J}|} = \frac{5}{\sqrt{10}}$$

Example 13.11 Find the distance from $X(3,1)$ to the line through $X_0(2,-3)$ and parallel to $\mathbf{V} = -\mathbf{I}+4\mathbf{J}$.

The vector $\mathbf{L}^\perp = -4\mathbf{I}-\mathbf{J}$ is orthogonal to the line. Thus the distance is

(13.30) $$|pr_{\mathbf{L}^\perp}(\mathbf{X}-\mathbf{X_0})| = \frac{|(\mathbf{I}+4\mathbf{J})\cdot(-4\mathbf{I}-\mathbf{J})|}{|-4\mathbf{I}-\mathbf{J}|} = \frac{8}{\sqrt{17}}$$

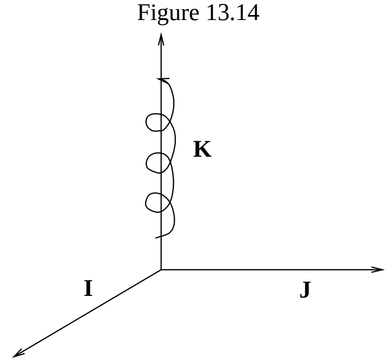
Example 13.12 Find the point on the line $L:\ 2x-3y=17$ which is closest to the origin.

Let $\mathbf{X}$ be the vector from the origin to the desired point. Then $\mathbf{X}$ is orthogonal to the line, so is parallel to the vector $\mathbf{L}^\perp = 2\mathbf{I}-3\mathbf{J}$. Writing $\mathbf{X} = t(2\mathbf{I}-3\mathbf{J})$, since $\mathbf{X}$ ends on the line we have $2(2t)-3(-3t) = 17$, so $t = 17/13$, and $\mathbf{X} = (34/13)\mathbf{I}-(51/13)\mathbf{J}$.

§13.3. Vectors in Space

In a Cartesian coordinate system for space, the vectors **I**, **J**, **K** are the vectors from the origin to the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ respectively. These are unit vectors, mutually orthogonal, and form the **standard base** for space. We always take a coordinatization so that $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ is a right-handed system. More precisely, if we situate **I** and **J** on the horizontal plane, then **I** is a unit vector, **J** is a unit vector perpendicular to **I** and counterclockwise from **I**, and **K** is a unit vector orthogonal to the horizontal plane, pointing upwards (see figure 13.14).

Figure 13.14



Any vector **V** can be written uniquely as

(13.31) $$\mathbf{V} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K},$$

where a, b, c are called the **components** of **V**. To add two vectors, add the components; to multiply a vector by a scalar, multiply the components by the scalar. If **V** is given as in (13.31), its **length** is

(13.32) $$|\mathbf{V}| = \sqrt{a^2 + b^2 + c^2}.$$

The **direction** of **V** is determined by the cosines of the angles between **V** and the coordinate axes. Thus, for any vector **V** we can write

(13.33) $$\mathbf{V} = |\mathbf{V}|(\cos\alpha\mathbf{I} + \cos\beta\mathbf{J} + \cos\gamma\mathbf{K})$$

where α, β, γ are those angles. The components of the unit vector in (13.33) are called the **direction cosines** of the vector **V**. Note that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition 13.7 *The* **dot product** *of two vectors* **V**, **W** *is defined as*

(13.34) $$\mathbf{V}\cdot\mathbf{W} = |\mathbf{V}||\mathbf{W}|\cos\theta,$$

where θ is the angle between **V** *and* **W**.

As for plane vectors, this has an easy formulation in terms of the components of the vectors.

Proposition 13.7 *Let*

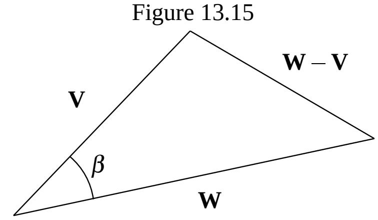
(13.35) $$\mathbf{V} = a_1\mathbf{I} + b_1\mathbf{J} + c_1\mathbf{K}, \quad \mathbf{W} = a_2\mathbf{I} + b_2\mathbf{J} + c_2\mathbf{K}$$

in components. Then

$$\mathbf{V}\cdot\mathbf{W} = a_1a_2 + b_1b_2 + c_1c_2 \tag{13.36}$$

To see this, we start with the Law of Cosines for the triangle whose sides are the vectors $\mathbf{V}$, $\mathbf{W}$, $\mathbf{W}-\mathbf{V}$ (see figure 13.15):

Figure 13.15



$|\mathbf{W}-\mathbf{V}|^2 = |\mathbf{V}|^2 + |\mathbf{W}|^2 - 2|\mathbf{V}||\mathbf{W}|\cos\beta$

$$|\mathbf{W}-\mathbf{V}|^2 = |\mathbf{W}|^2 + |\mathbf{V}|^2 - 2|\mathbf{V}||\mathbf{W}|\cos\beta = |\mathbf{W}|^2 + |\mathbf{V}|^2 - 2(\mathbf{W}\cdot\mathbf{V})\ , \tag{13.37}$$

so that

$$\mathbf{W}\cdot\mathbf{V} = \frac{1}{2}(|\mathbf{W}|^2 + |\mathbf{V}|^2 - |\mathbf{W}-\mathbf{V}|^2)\ . \tag{13.38}$$

Now, writing the right hand side in terms of components, using (13.31) and (13.32), we get (13.36), after some cancellation. In particular, just as in two dimensions, two vectors $\mathbf{V}$, $\mathbf{W}$ are **orthogonal** if

$\mathbf{V}\cdot\mathbf{W} = 0$.

Example 13.13 Find the angle between the vectors $\mathbf{V} = 2\mathbf{I} - 3\mathbf{J} + \mathbf{K}$, $\mathbf{W} = 6\mathbf{I} + \mathbf{J} - 2\mathbf{K}$.

We have $\mathbf{V}\cdot\mathbf{W} = 12 - 3 - 2 = 7$ and $|\mathbf{V}| = \sqrt{2^2+3^2+1^2} = 3.74$, $|\mathbf{W}| = \sqrt{6^2+1^2+2^2} = 6.40$. Thus

$$\cos\alpha = \frac{7}{(3.74)(6.40)} = .2923 \tag{13.39}$$

so $\alpha = 73°$.

Example 13.14 Find a vector orthogonal to both the vectors $\mathbf{V}$ and $\mathbf{W}$ of example 13.13.

Let $\mathbf{X} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ be the desired vector. We have the conditions

$$\mathbf{X}\cdot\mathbf{V} = 2x - 3y + z = 0\ , \qquad \mathbf{X}\cdot\mathbf{W} = 6x + y - 2z = 0\ . \tag{13.40}$$

We can solve these equations by replacing z by any nonzero value, say $z = 1$, and solving the resulting equations for x and y:

$$2x - 3y + 1 = 0\ , \qquad 6x + y - 2 = 0\ . \tag{13.41}$$

These have the solution $x = 1/4,\ y = 1/2$. Thus we can take

$$\mathbf{X_0} = \frac{1}{4}\mathbf{I} + \frac{1}{2}\mathbf{J} + \mathbf{K} \tag{13.42}$$

as our answer. Of course there is a line of such vectors, corresponding to all possible values for z. Thus the set of all vectors orthogonal to $\mathbf{V}$ and $\mathbf{W}$ is the set $\{t\mathbf{X_0}\}$.

Given vectors $\mathbf{V}$ and $\mathbf{W}$, the **projection** of $\mathbf{V}$ (denoted $pr_{\mathbf{W}}(\mathbf{V})$) in the direction of $\mathbf{W}$ is the vector $\mathbf{V}'$ parallel to $\mathbf{W}$ such that $\mathbf{V}$ and $\mathbf{V} - \mathbf{V}'$ are orthogonal. If β is the angle between $\mathbf{V}$ and $\mathbf{W}$, this projection is the vector of length $|\mathbf{V}|\cos\beta$ in the direction of $\mathbf{W}$. The formula for the projection is (as in the plane):

$$pr_{\mathbf{W}}(\mathbf{V}) = (\frac{\mathbf{V}\cdot\mathbf{W}}{\mathbf{W}\cdot\mathbf{W}})\mathbf{W}\ . \tag{13.43}$$

Again, just as in the plane, if $\mathbf{U}$ is the unit vector in the direction of $\mathbf{W}$, then $pr_{\mathbf{W}}(\mathbf{V}) = (\mathbf{V}\cdot\mathbf{U})\mathbf{U}$. We note that for two vectors $\mathbf{V_1},\ \mathbf{V_2}$,

$$pr_{\mathbf{W}}(\mathbf{V_1} + \mathbf{V_2}) = pr_{\mathbf{W}}(\mathbf{V_1}) + pr_{\mathbf{W}}(\mathbf{V_2})\ . \tag{13.44}$$

Definition 13.8 *The* **cross product** *of two vectors* $\mathbf{V},\ \mathbf{W}$*, denoted* $\mathbf{V}\times\mathbf{W}$*, is that vector*
a) of length the area of the parallelogram spanned by $\mathbf{V},\ \mathbf{W}$*,*
b) perpendicular to the plane of $\mathbf{V},\ \mathbf{W}$ *so that the system* $\{\mathbf{V},\ \mathbf{W},\ \mathbf{V}\times\mathbf{W}\}$ *is right-handed.*

Now, since the area of the parallelogram spanned by the vectors $\mathbf{V},\ \mathbf{W}$ is $|\mathbf{V}||\mathbf{W}|\sin\beta$, where β is the angle between the two vectors, we have

$$|\mathbf{V}\times\mathbf{W}|^2 = |\mathbf{V}|^2|\mathbf{W}|^2 - (\mathbf{V}\cdot\mathbf{W})^2 \tag{13.45}$$

since

$$|\mathbf{V}\times\mathbf{W}|^2 = |\mathbf{V}|^2|\mathbf{W}|^2\sin^2\beta = |\mathbf{V}|^2|\mathbf{W}|^2(1-\cos^2\beta) = |\mathbf{V}|^2|\mathbf{W}|^2 - (|\mathbf{V_1}||\mathbf{V_2}|\cos\beta)^2 \tag{13.46}$$

which is the right side of (13.45), from (13.34). Note that interchanging $\mathbf{V}$ and $\mathbf{W}$ changes the sign of the

cross product, for if the system $\{\mathbf{V},\ \mathbf{W},\ \mathbf{L}\}$ is right-handed, then the system $\{\mathbf{W},\ \mathbf{V},\ \mathbf{L}\}$ is left-handed, and thus $\{\mathbf{W},\ \mathbf{V},\ -\mathbf{L}\}$ is right handed. This gives us the first of the following identities:

$$\mathbf{V}\times\mathbf{W} = -\mathbf{W}\times\mathbf{V} \tag{13.47}$$

$$\mathbf{V}\times\mathbf{V} = \mathbf{0}, \tag{13.48}$$

$$(a\mathbf{V})\times\mathbf{W}) = a(\mathbf{V}\times\mathbf{W}), \tag{13.49}$$

We now determine a formula for the cross product in components. It is useful to start with the determinant of three vectors in space, sometimes called the **triple scalar product**.

Definition 13.9 *Given three vectors in space* $\mathbf{U},\ \mathbf{V},\ \mathbf{W}$*, we define the* **determinant** $\det(\mathbf{U},\mathbf{V},\mathbf{W})$ *as the signed volume of the parallelepiped spanned by the vectors. This is zero if the vectors all lie in the same* *plane. Otherwise, the sign is positive if the vectors* $\{\mathbf{U}, \mathbf{V}, \mathbf{W}\}$ *form a right-handed system, and negative if a left-handed system.*

Proposition 13.8 *Given two vectors* $\mathbf{V}$, $\mathbf{W}$*, then, for any third vector* $\mathbf{U}$,

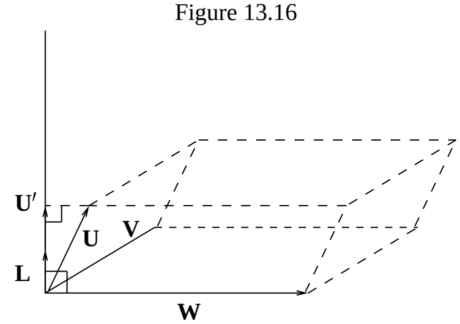
(13.50) $$\det(\mathbf{U}, \mathbf{V}, \mathbf{W}) = \mathbf{U} \cdot (\mathbf{V} \times \mathbf{W})$$

For any two vectors $\mathbf{U_1}$, $\mathbf{U_2}$

(13.51) $$\det(\mathbf{U_1} + \mathbf{U_2}, \mathbf{V}, \mathbf{W}) = \det(\mathbf{U_1}, \mathbf{V}, \mathbf{W}) + \det(\mathbf{U_2}, \mathbf{V}, \mathbf{W})$$

We now show (13.50) using a geometric argument similar to that used for proposition 13.4. If $\mathbf{V}$ and $\mathbf{W}$ lie on a line, then all terms are zero, and there is nothing to show. Otherwise, $\mathbf{V}$ and $\mathbf{W}$ determine a plane; let $\mathbf{L}$ be the unit vector orthogonal to that plane so that the triple $\mathbf{V}, \mathbf{W}, \mathbf{L}$ is right-handed. For any vector $\mathbf{U}$, let $\mathbf{U}'$ be the projection of $\mathbf{U}$ in the direction of $\mathbf{L}$. Then, we see geometrically that the volume of the parallelepiped spanned by $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is the product of the area of the parallelogram spanned by $\mathbf{V}, \mathbf{W}$ and the length of $\mathbf{U}'$ (see figure 13.16).

Figure 13.16



Since $\mathbf{V} \times \mathbf{W}$ has the same direction as $\mathbf{L}$, this volume is

(13.52) $$|\mathbf{U}'||\mathbf{V} \times \mathbf{W}| = |\mathbf{U}||\mathbf{V} \times \mathbf{W}| \cos\beta = \mathbf{U} \cdot (\mathbf{V} \times \mathbf{W})$$

where β is the angle between $\mathbf{U}$ and $\mathbf{L}$. The signs are right in (21), for on both sides they are determined by whether or not the system $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is right-handed. 2 now follows directly from (21), since the right hand side is linear in $\mathbf{U}$:

(13.53) $$\det(\mathbf{U_1} + \mathbf{U_2}, \mathbf{V}, \mathbf{W}) = (\mathbf{U_1} + \mathbf{U_2}) \cdot (\mathbf{V} \times \mathbf{W})$$

(13.54) $$= \mathbf{U_1} \cdot (\mathbf{V} \times \mathbf{W}) + \mathbf{U_2} \cdot (\mathbf{V} \times \mathbf{W}) = \det(\mathbf{U_1}, \mathbf{V}, \mathbf{W}) + \det(\mathbf{U_2}, \mathbf{V}, \mathbf{W})$$

Now, if we permute the three vectors $\mathbf{U}$. $\mathbf{V}$, $\mathbf{W}$, we just change the sign of the determinant, since it is always the parallelepiped spanned by the same vectors:

(13.55) $$\det(\mathbf{U}, \mathbf{V}, \mathbf{W}) = -\det(\mathbf{V}, \mathbf{U}, \mathbf{W}) = \det(\mathbf{V}, \mathbf{W}, \mathbf{U})\ .$$

So, since, but for sign, we can move any of the vectors in $\det(\mathbf{U}, \mathbf{V}, \mathbf{W})$ to the first position, we conclude that the determinant is linear in all three variables. In particular, the cross product is linear in its variables.

This allows us to calculate the determinant and cross product from the components of the given vectors. We first observe that the calculations for the basis vectors are immediate, since the area of the unit square is 1:

$$\mathbf{I}\times\mathbf{J}=\mathbf{K},\ \mathbf{J}\times\mathbf{K}=\mathbf{I},\ \mathbf{K}\times\mathbf{I}=\mathbf{J},\ \mathbf{I}\times\mathbf{I}=\mathbf{J}\times\mathbf{J}=\mathbf{K}\times\mathbf{K}=\mathbf{0} \tag{13.56}$$

Finally, from (21) $\mathbf{J}\times\mathbf{I}=-\mathbf{I}\times\mathbf{J}=-\mathbf{K}$, etc. After a long computation, we find:

Proposition 13.9 *If* $\mathbf{V_1}=a_1\mathbf{I}+b_1\mathbf{J}+c_1\mathbf{K},\ \mathbf{V_2}=a_2\mathbf{I}+b_2\mathbf{J}+c_2\mathbf{K}$, *then*

$$\mathbf{V_1}\times\mathbf{V_2}=(b_1c_2-c_1b_2)\mathbf{I}+(c_1a_2-a_1c_2)\mathbf{J}+(a_1b_2-b_1a_2)\mathbf{K}. \tag{13.57}$$

Now we see that the determinant of three vectors, or, what is the same, the triple scalar product: $\mathbf{V_1}\cdot(\mathbf{V_2}\times\mathbf{V_3})=(\mathbf{V_1}\times\mathbf{V_2})\cdot\mathbf{V_3}$ is, in fact, the determinant of the matrix whose rows are the components of the vectors $\mathbf{V_1},\mathbf{V_2},\mathbf{V_3}$, just by taking the dot product of $\mathbf{V_3}$ with the expression (13.57) for $\mathbf{V_1}\times\mathbf{V_2}$:

Proposition 13.10 *If, in addition,* $\mathbf{V_3}=a_3\mathbf{I}+b_3\mathbf{J}+c_3\mathbf{K}$, *then*

$$\det(\mathbf{V_1},\mathbf{V_2},\mathbf{V_3})=(\mathbf{V_1}\times\mathbf{V_2})\cdot\mathbf{V_3}=a_3(b_1c_2-c_1b_2)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)\ . \tag{13.58}$$

This is just the expansion of the determinant by minors of the third row. An easy way to remember the formula for the cross product is as this determinant:

$$\mathbf{V_1}\times\mathbf{V_2}=\det\begin{pmatrix}\mathbf{I}&\mathbf{J}&\mathbf{K}\\ a_1&b_1&c_1\\ a_2&b_2&c_2\end{pmatrix} \tag{13.59}$$

Example 13.15 Find $\mathbf{V_1}\cdot(\mathbf{V_2}\times\mathbf{V_3})$ where

$$\mathbf{V_1}=-\mathbf{I}+2\mathbf{J}+\mathbf{K}.\quad \mathbf{V_2}=2\mathbf{I}-2\mathbf{J}+3\mathbf{K},\quad \mathbf{V_3}=\mathbf{I}-2\mathbf{K}\ . \tag{13.60}$$

By proposition 13.9, this is the determinant

$$\mathbf{V_1}\cdot(\mathbf{V_2}\times\mathbf{V_3})=\det\begin{pmatrix}-1&2&1\\ 2&-2&3\\ 1&0&-2\end{pmatrix}=1(6+2)+0+(-2)(2-4)=12 \tag{13.61}$$

where we calculate by minors of the third row.

Example 13.16 Find a vector $\mathbf{W}$ of length $|\mathbf{W}|=5$ which is orthogonal to both $\mathbf{V_1}$ and $\mathbf{V_2}$, so that the system $\{\mathbf{V_1},\ \mathbf{V_2},\ \mathbf{W}\}$ is right-handed.
$\mathbf{W}$ is a positive multiple of $V_1\times V_2$, which is

$$\mathbf{V_1}\times\mathbf{V_2}=\det\begin{pmatrix}\mathbf{I}&\mathbf{J}&\mathbf{K}\\ -1&2&1\\ 2&-2&3\end{pmatrix}=8\mathbf{I}+5\mathbf{J}-2\mathbf{K}\ . \tag{13.62}$$

This vector has length $\sqrt{64+25+4}=\sqrt{93}$, so

$$\mathbf{W}=\frac{5}{\sqrt{93}}(8\mathbf{I}+5\mathbf{J}-2\mathbf{K})\ . \tag{13.63}$$

§13.4. Lines and Planes in Space

A coordinate system in space consists of a choice of a particular point O as origin, and a right-handed system of mutually orthogonal unit vectors $\mathbf{I}$, $\mathbf{J}, \mathbf{K}$. Once a coordinate system is selected, we can represent a point $P : (x, y, z)$ by the vector $\vec{OP} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ from the origin to P. Given another point $Q = (x', y', z')$, the *vector* from P to Q is denoted $\vec{PQ} = (x' - x)\mathbf{I} + (y' - y)\mathbf{J} + (z' - z)\mathbf{K}$. We shall often write the vector $\vec{OP}$ as $\mathbf{P}$ for consistency of notation in formaluas. The **line** through a given point P and

in the direction of a given vector $\mathbf{L}$ is the set of all points X of the form

(13.64) $$\mathbf{X} = \mathbf{P} + t\mathbf{L}$$

where t runs over all real numbers. This is called the **parametric form** of the equation of the line. This says that the vector $\mathbf{X} - \mathbf{P}$ is collinear with the vector $\mathbf{L}$, and thus the components are proportional. In coordinates, writing $\mathbf{X} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$, $\mathbf{P} = x_0\mathbf{I} + y_0\mathbf{J} + z_0\mathbf{K}$, and $\mathbf{L} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$, we get the **symmetric form** of the equation of a line:

(13.65) $$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} .$$

Example 13.17 Find the symmetric equations of the line through the points $P(2,-1,4)$ and $Q(6, 2, -3)$.

The vector $\vec{PQ} = 4\mathbf{I} + 3\mathbf{J} - 7\mathbf{K}$ is on the line, so $\mathbf{X} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ is on the line precisely when $\mathbf{X} - \mathbf{P}$ is parallel to $\vec{PQ}$. This gives us the symmetric equations

(13.66) $$\frac{x - 2}{4} = \frac{y + 1}{3} = \frac{z - 4}{-7} .$$

The **plane** through a point P, **spanned** by the vectors $\mathbf{V}$ and $\mathbf{W}$ is the set of all points X of the form

(13.67) $$\mathbf{X} = \mathbf{P} + s\mathbf{V} + t\mathbf{W}$$

where s, t range over all real numbers. This is the **parametric form** of a plane. We note that a point X is on the plane if and only if the parallelipiped formed from $\mathbf{X} - \mathbf{P}$, $\mathbf{V}$, $\mathbf{W}$ has zero volume, that is

(13.68) $$\det(\mathbf{X} - \mathbf{P}, \mathbf{V}, \mathbf{W}) = \mathbf{0} .$$

This is the equation of the plane. The vector $\mathbf{N} = \mathbf{V} \times \mathbf{W}$ is called the **normal** to the plane, since it is orthogonal to all vectors lying on the plane. In terms of the normal, we have this as the equation of the plane:

(13.69) $$(\mathbf{X} - \mathbf{P}) \cdot \mathbf{N} = 0 ,$$

since $\det(\mathbf{X} - \mathbf{P}, \mathbf{V}, \mathbf{W}) = (\mathbf{X} - \mathbf{P}) \cdot \mathbf{N}$. Turning to coordinates, let P be the point (x_0, y_0, z_0), and $\mathbf{N} = a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. Then for (x, y, z) the coordinates for the point X, (13.69) becomes

(13.70) $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d .$$

where

(13.71) $$d = ax_0 + by_0 + cz_0 .$$

We can summarize this discussion with

Proposition 13.11 *a) Given a point P and a vector* **N**, *the plane through P and orthogonal to* **N** *is given by the equation*

$$\mathbf{X}\cdot\mathbf{N}=\mathbf{P}\cdot\mathbf{N}\ . \tag{13.72}$$

b) The plane through P spanned by **V** *and* **W** *has as normal* $\mathbf{N}=\mathbf{V}\times\mathbf{W}$.
c) The coefficients of the cartesian equation (13.70) for a plane are the components of the normal vector.

Example 13.18 Find the equation of the plane through the point P(5,3,-1) perpendicular to the line in space whose symmetric equations are

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-1}{-2} \tag{13.73}$$

The vector $3\mathbf{I}+4\mathbf{J}-2\mathbf{K}$ has the direction of the line, so is normal to the plane, and can be taken to be **N**. We know that the equation of the plane has the form $(\mathbf{X}-\mathbf{P})\cdot\mathbf{N}=0$, for $P:(5,3,-1)$ is a point on the plane. This gives the equation

$$\mathbf{X}\cdot\mathbf{N}=\mathbf{P}\cdot\mathbf{N}\quad\text{or}\quad 3x+4y-2z=15+12+2=29\ . \tag{13.74}$$

Example 13.19 Find the equation of the plane containing the points $P(2,5,-1)$, $Q(6,-1,0)$, $R(3,1,4)$.

The vectors $\vec{PQ}=4\mathbf{I}-6\mathbf{J}+\mathbf{K}$, $\vec{PR}=\mathbf{I}-4\mathbf{J}+5\mathbf{K}$ lie on the plane, so the normal is

$$\mathbf{N}=\vec{PQ}\times\vec{PR}=(-30+4)\mathbf{I}+(1-20)\mathbf{J}+(-16+6)\mathbf{K}=-26\mathbf{I}-19\mathbf{J}-10\mathbf{K}\ . \tag{13.75}$$

The equation of the plane then is $\mathbf{X}\cdot\mathbf{N}=\mathbf{P}\cdot\mathbf{N}$, which comes to $26x+19y+10z=137$.

Example 13.20 Find the equation of the line through the origin and orthogonal to the plane $2x-y+3z=1$.

The vector $2\mathbf{I}-\mathbf{J}+3\mathbf{K}$ is normal to the plane, so lies in the direction of the line.Thus the symmetric equations of the line are

$$\frac{x}{2}=\frac{y}{-1}=\frac{z}{3} \tag{13.76}$$

Now, given two planes with equations $\mathbf{X}\cdot\mathbf{N_1}=d_1$, $\mathbf{X}\cdot\mathbf{N_2}=d_2$, the vector $\mathbf{N_1}\times\mathbf{N_2}$ has the direction of the line of intersection of the two planes. Thus if P is a point on that line (found by finding a simultaneous solution of the equations of the planes), the equation of the line is

$$\mathbf{X}=\mathbf{P}+t(\mathbf{N_1}\times\mathbf{N_2})\ . \tag{13.77}$$

Example 13.21 Find the parametric form of the line given by the equations $2x-y+3z=1$, $x+5y-2z=0$.

To find a point P on the line we solve the simultaneous equations, taking $z = 0$. This gives the equations for x and y: $2x - y = 1$, $x + 5y = 0$. The solution is $x = 20/11$, $y = -4/11$. Thus $P(20/11, -4/11, 0)$ is on the line. The cross product of the two normals is

$$(2\mathbf{I} - 1\mathbf{J} + 3\mathbf{K}) \times (\mathbf{I} + 5\mathbf{J} - 2\mathbf{K}) = -13\mathbf{I} + 7\mathbf{J} + 12\mathbf{K} \tag{13.78}$$

giving the parametric equation of the line

$$\mathbf{X} = \left(\frac{20}{11} - 13t\right)\mathbf{I} + \left(\frac{-4}{11} + 7t\right)\mathbf{J} + 12t\mathbf{K} \tag{13.79}$$

Now, suppose we are given two lines in parametric form:

$$\mathbf{X} = \mathbf{P_1} + t\mathbf{L_1}\ , \quad \mathbf{X} = \mathbf{P_2} + t\mathbf{L_2}\ , \tag{13.80}$$

and a point Q, and are asked to find the equation of the plane through Q and parallel to the lines. Then the normal to this plane is perpendicular to the two lines, so can be taken to be $\mathbf{L_1} \times \mathbf{L_2}$, and then the equation of the desired plane is

$$(\mathbf{X} - \mathbf{Q}) \cdot (\mathbf{L_1} \times \mathbf{L_2}) = 0\ . \tag{13.81}$$

Example 13.22 Find the equation of the plane through a (2,0,-1) parallel to the vectors $\mathbf{V} = 2\mathbf{I} - \mathbf{J}$, $\mathbf{W} = 6\mathbf{I} + \mathbf{K}$.

$\mathbf{V} \times \mathbf{W}$ is perpendicular to the vectors $\mathbf{V}, \mathbf{W}$, so can be taken as the normal $\mathbf{N}$ to the plane. We get

$$\mathbf{N} = (2\mathbf{I} - \mathbf{J}) \times (6\mathbf{I} + \mathbf{K}) = 2\mathbf{I} \times \mathbf{K} - 6\mathbf{J} \times \mathbf{I} - \mathbf{J} \times \mathbf{K} = -2\mathbf{J} + 6\mathbf{K} - \mathbf{I}\ . \tag{13.82}$$

Taking $\mathbf{X_0} = 2\mathbf{I} - \mathbf{K}$ as a given point on the plane, the equation $\mathbf{X} \cdot \mathbf{N} = \mathbf{X_0} \cdot \mathbf{N}$ is

$$-x - 2y + 6z = 2(-1) + (-1)6 = -8 \tag{13.83}$$

We can summarize this discussion in the form of two assertions.

Proposition 13.12 *a) Given a line* $\mathbf{X} = \mathbf{P} + t\mathbf{L}$*, the plane through a given point Q and perpendicular to the line has the equation* $(\mathbf{X} - \mathbf{Q}) \cdot \mathbf{L} = 0$*.*
b) Given the equation of a plane $\mathbf{X} \cdot \mathbf{N} = d$*, a point P. the line through P and perpendicular to the plane has the equation* $\mathbf{X} = \mathbf{P} + t\mathbf{N}$*.*

Now, suppose we want to find the distance of a point Q to a plane Π. We know from elementary geometry that the this distance is the length of the line segment from Q to Π which is perpendicular to Π. This line segment is thus in the direction of the normal to Π, and is seen (see figure 13.16) to be the projection of any vector from Q to Π in the normal direction. This demonstrates the first part of

Proposition 13.13 *a) The distance from a point Q to a plane* Π *with normal* $\mathbf{N}$ *is*

$$d(Q, \Pi) = \frac{|\vec{PQ} \cdot \mathbf{N}|}{|\mathbf{N}|}\ , \tag{13.84}$$

where P is any point on the plane. b) The distance from a point Q to a line L in the direction **L** *is*

$$d(Q,L) = \frac{|\vec{PQ} \times \mathbf{L}|}{|\mathbf{L}|} \ , \tag{13.85}$$

where P is any point on the line.

To show b), start with figure 13.17. We have

$$d(Q,L) = |\vec{PQ}| \sin\theta = \frac{|\vec{PQ}||\mathbf{L}| \sin\theta}{|\mathbf{L}|} = \frac{|\vec{PQ} \times \mathbf{L}|}{|\mathbf{L}|} \ . \tag{13.86}$$

Figure 13.17

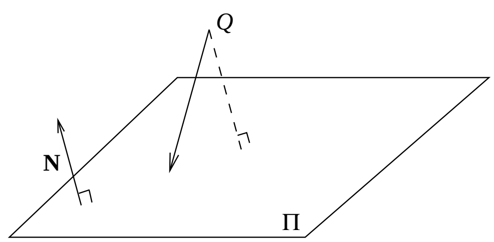
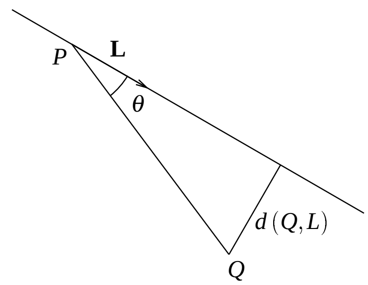


Figure 13.18



Example 13.23 Find the distance of the point $(2,0,4)$ from the plane whose equation is $x+y-2z=0$.

Let $Q:(2,0,4)$. Pick a point P on the plane, for example, $P=(1,1,1)$. $\mathbf{N}=\mathbf{I}+\mathbf{J}-2\mathbf{K}$ is normal to the plane, so the distance is the length of the projection of the vector from P to Q in the direction of $\mathbf{N}$:

$$\vec{PQ} \cdot \mathbf{N} = (\mathbf{I}-\mathbf{J}+3\mathbf{K}) \cdot (\mathbf{I}+\mathbf{J}-2\mathbf{K}) = -6 \ , \quad |\mathbf{N}| = \sqrt{6} \tag{13.87}$$

so the distance is $|\vec{PQ} \cdot \mathbf{N}|/|\mathbf{N}| = \sqrt{6}$.

Example 13.24 Find the distance of the point $(2,0,1)$ from the line whose symmetric equations are

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-1}{-2} \tag{13.88}$$

Let $\vec{OQ} = 2\mathbf{I}+\mathbf{K}$ be the vector to the given point, and $\vec{OP} = 2\mathbf{I}-\mathbf{J}+\mathbf{K}$ the vector to a point on the line, and $\mathbf{L} = 3\mathbf{I}+4\mathbf{J}-2\mathbf{K}$, a vector in the direction of the line. The distance is

$$\frac{|\vec{PQ} \times \mathbf{L}|}{|\mathbf{L}|} = \frac{|-\mathbf{J} \times (3\mathbf{I}+4\mathbf{J}-2\mathbf{K})|}{|3\mathbf{I}+4\mathbf{J}-2\mathbf{K}|} = \sqrt{\frac{13}{29}} \ . \tag{13.89}$$

Example 13.25 Find the distance between the two parallel planes

$$\Pi_1 : \quad x+2y-5z = 2 \ , \qquad \Pi_2 : \quad x+2y-5z = 11 \ . \tag{13.90}$$

The distance between the two planes is the length of any line segment perpendicular to both planes. Thus we need only find the length of the projection of $\vec{P_1 P_2}$ on the common normal $\mathbf{N} = \mathbf{I} + 2\mathbf{J} - 5\mathbf{K}$, where P_1 is a point on Π_1 and P_2 is a point on Π_2. Since $\mathbf{P_1} \cdot \mathbf{N} = 2$, and $\mathbf{P_2} \cdot \mathbf{N} = 11$ for these points we get, for the distance:

$$\frac{|(\mathbf{P_2} - \mathbf{P_1}) \cdot \mathbf{N}|}{|\mathbf{N}|} = \frac{11 - 2}{\sqrt{1 + 4 + 25}} = \frac{9}{\sqrt{30}} . \tag{13.91}$$