Calculus III
Practice Problems 9: Answers

1. Find the surface area of the part of the hyperbolic paraboloid $z = xy$ that lies inside the cylinder $x^2 + y^2 \leq 4$.

Answer. Let R be the disk $x^2 + y^2 \leq 4$. We want the surface area of the part of the surface $z = xy$ lying over R. Now $z_x = y,\ z_y = x$, so that

$$Surface\ \ Area = \int\int_R dS = \int\int_R \sqrt{1 + z_x^2 + z_y^2}dxdy = \int\int_R \sqrt{1 + y^2 + x^2}dA\ .$$

It is convenient to switch to polar coordinates. We get

$$Surface\ Area = \int_0^{2\pi}\int_0^2 \sqrt{1 + r^2}rdrd\theta = \frac{2\pi}{3}(5^{3/2} - 1)\ .$$

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2. Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Answer. The surface lies over the region $R = \{1 \leq x^2 + y^2 \leq 4\}$. Since $z_x = -2x,\ z_y = 2y,\ dS = \sqrt{1 + 4x^2 + 4y^2}dA$. Now, switching to polar coordinates, the surface area is

$$\int_0^{2\pi}\int_1^2 \sqrt{1 + 4r^2}rdrd\theta = 2\pi\left[\frac{1}{2}\frac{1}{8}\frac{2}{3}(1 + 4r^2)^{\frac{3}{2}}\right]\Big|_1^2 = \frac{1}{12}[17^{\frac{3}{2}} - 5^{\frac{3}{2}}]\ .$$

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3. Find the surface area of the part of the surface $z = (2/3)(x^{3/2} + y^{3/2})$ that lies above the triangle in the first quadrant bounded by the line $x + y = 1$.

Answer. We use the formula $dS = \sqrt{1 + z_x^2 + z_y^2}dxdy$:

$$\frac{\partial z}{\partial x} = \sqrt{x}\ , \quad \frac{\partial z}{\partial y} = \sqrt{y}\ ,$$

so $dS = \sqrt{1 + x + y}dxdy$. The region is the type 1 domain given by the inequalities $0 \leq x \leq 1,\ 0 \leq y \leq 1 - x$, so

$$S = \int_0^1 [\int_0^{1-x} \sqrt{1 + x + y}dy]dx = \int_0^1 [\frac{2}{3}(1 + x + y)^{3/2}|_0^{1-x}]dx = \int_0^1 \frac{2}{3}2^{3/2}dx = \frac{2}{3}\sqrt{8}\ .$$

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4. Find the mass and the x-coordinate of the center of mass of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$ with the density function $\rho(x, y, z) = y$.

Answer. The solid can be represented by the inequalities, $0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y$. Thus

$$Mass = \int\int\int \rho dV = \int_0^1 [\int_0^{1-x} [\int_0^{1-x-y} ydz]dy]dx\ .$$

The innermost integral is $y(1 - x - y) = y(1 - x) - y^2$. Then

$$\int_0^{1-x} (y(1 - x) - y^3)dy = [\frac{y^2}{2}(1 - x) - \frac{y^3}{3}]_0^{1-x} = \frac{(1 - x)^3}{6}\ ,$$

and the mass is

$$Mass\ = \frac{1}{6}\int_0^1 (1 - x)^3 dx = \frac{1}{24}\ .$$

Now

$$Mom_{x=0} = \int_0^1 [\int_0^{1-x} [\int_0^{1-x-y} xydz]dy]dx \ .$$

The innermost integral is $xy(1-x-y) = x[y(1-x)-y^2]$, and

$$\int_0^{1-x} (x[y(1-x)-y^3)]dy = x\big[\frac{y^2}{2}(1-x) - \frac{y^3}{3}\big]_0^{1-x} = \frac{x(1-x)^3}{6} \ .$$

Finally,

$$Mom_{x=0} = \frac{1}{6}\int_0^1 x(1-x)^3 dx = \frac{1}{6}\int_0^1 (u^3-u^4)du = \frac{1}{120} \ ,$$

using the substitution $u = 1-x$. Thus $\bar{x} = (1/120)/(1/24) = 1/5$.

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5. Find the center of mass of the piece of the solid parabolic shell $z \le 16-(x^2+y^2)$ lying above the xy-plane.

Answer. Since this is a solid of revolution about the z-axis, the center of mass lies on the z axis, so is of the form $(0,0,\bar{z})$. To find $\bar{z}$ we must calculate the volume and $Mom_{z=0}$ for the region. For these calculations we switch to cylindrical coordinates.

$$Volume = \int_0^{2\pi}\int_0^4 (16-r^2)rdrd\theta = 2\pi\int_0^4 (16r-r^3)dr = 2\pi(8r^2-\frac{r^4}{4})_0^4 = 2^7\pi \ .$$

$$Mom_{z=0} = \int_0^{2\pi}\int_0^4 (\int_0^{16-r^2} zdz)rdrd\theta = \pi\int_0^4 (16-r^2)^2 rdrd\theta \ .$$

We conclude the computation with the change of variable $u = 16-r^2, \ du = -2rdr$:

$$Mom_{z=0} = \pi\int_0^{16} u^2du = \frac{2^{12}\pi}{6} \ .$$

Thus

$$\bar{z} = \frac{1}{6}\frac{2^{12}\pi}{2^7\pi} = \frac{32}{6} = 5.333 \ .$$

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6. Find the average value of $f(x,y,z) = x+y+z$ over the region R in the first octant (the region where all the coordinates are positive) under the plane $x+y+z = 1$.

Answer. R can be described as the set of (x,y,z) satisfying $0 \le x \le 1, \ 0 \le y \le 1-x, \ 0 \le z \le 1-x-y$. This description tells us how to calculate by iterated integrals:

$$Volume = \int\int\int_R dV = \int_0^1 [\int_0^{1-x} [\int_0^{1-x-y} dz]dy]dx \ ,$$

$$Total = \int\int\int_R fdV = \int_0^1 [\int_0^{1-x} [\int_0^{1-x-y} (x+y+z)dz]dy]dx \ ,$$

and the average is $Total/Volume$. The computations are tedious and involve only integrations of polynomials. The final integrations (with respect to x) are

$$Volume = \int_0^1 \frac{(1-x)^2}{2}dx = \frac{1}{6} \ ,$$

$$Total = \int_0^1 (\frac{1}{3} - \frac{x}{2} + \frac{x^3}{6})dx = \frac{1}{8} \ .$$

Thus the average is (1/8)/(1/6) = 3/4.

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7. The curve $z = (x-1)^2, 0 \leq z \leq 1$ is rotated about the z-axis, enclosing, together with the xy-plane, a 3-dimensional region R. R is filled with a substance whose density is inversely proportional to the distance from the z-axis. Find the total mass of this object.

Answer. The region is that under the curve (using polar coordinates) $z = (r-1)^2$ above the disc $R = \{0 \leq r \leq 1\}$ on the xy-plane. The density is $\delta = k/r$. Thus, the mass is

$$\int\int_R z\delta dr d\theta = \int_0^{2\pi} \Big[\int_0^1 (r-1)^2 \frac{k}{r} r dr\Big] d\theta = 2\pi k \frac{(r-1)^3}{3}\Big|_0^1 = \frac{2\pi k}{3}$$

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8. Evaluate

$$\int\int\int_R (x^2+y^2+z^2) dx dy dz$$

where R is the ball $x^2+y^2+z^2 \leq 4$.

Answer. Switch to spherical coordinates. R is given by $\rho \leq 2$, and the integral is

$$\int\int\int_R \rho^2 dV = \int_0^{\pi}\int_0^{2\pi}\int_0^2 \rho^4 \sin\phi \, d\rho d\theta d\phi = 4\pi\frac{2^5}{5} \, .$$

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9. Find the centroid of the region R described in example 24, Chapter 17.

Answer. There we found $Volume = 60$. We used the change of variables

$$u = x+y+z\,, \quad v = y+z\,, \quad w = z$$

$$x = u-v\,, \quad y = v-w\,, \quad z = w$$

to do the computation. We had the Jacobian equal to 1, so it is easy to do the computation in the u, v, w variables:

$$Mom_{\{x=0\}} = \int\int\int_R x dV = \int -0^5 \int_0^3 \int_0^4 (u-v) du dv dw = 4\int_0^5 \Big[\int_0^3 (u-v) dv\Big] du \, .$$

The inner integral is

$$(uv - \frac{v^2}{2})\Big|_0^3 = 3u - \frac{9}{2} \, .$$

Thus

$$Mom_{\{x=0\}} = 4\int_0^5 (3u - \frac{9}{2}) du = 4\frac{5}{2}(15-9) = 60 \, .$$

$$Mom_{\{y=0\}} = \int\int\int_R y dV = \int -0^5 \int_0^3 \int_0^4 (v-w) du dv dw = 5\int_0^4 \Big[\int_0^3 (v-w) dv\Big] dw \, .$$

$$Mom_{\{y=0\}} = 5\int_0^4 (\frac{9}{2} - 3w) dw = \frac{5}{2}(9w - 3w^2)\Big|_0^4 = -30 \, .$$

Finally

$$Mom_{\{z=0\}} = \int\int\int_R z dV = \int -0^5 \int_0^3 \int_0^4 w du dv dw = 15\int_0^4 w dw = 120 \, .$$

Thus the centroid is at (1,-1/2,2). For completeness, here is **Example 24**. Find the volume of the region R

given by the inequalities

$$0 \le z \le 4\ , \quad 0 \le y+z \le 3\ , \quad 0 \le x+y+z \le 5\ .$$

This region is a parallelipiped, so by the appropriate change of coordinates, can be made to correspond to a rectangular parallelipiped. That is, we make the change of variables

$$u = x+y+z\ , \quad v = y+z\ , \quad w = z$$

so that R corresponds to the region S given by the inequalities $0 \le u \le 5,\ 0 \le v \le 3,\ 0 \le w \le 4$. Thus

$$Volume = \int\int\int_R dxdydz = \int\int\int_S \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw\ .$$

Now, to calculate the Jacobian, we solve for x, y, z in terms of u, v, w:

$$x = u - v\ , \quad y = v - w\ , \quad z = w\ ,$$

so that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1\ .$$

Thus

$$Volume = \int_0^5 \int_0^3 \int_0^4 dudvdw = 60\ .$$