

Calculus III
Practice Problems 7: Answers

1. What is the volume of the region under the surface $z = \ln x + y$ and over the rectangle in the xy -plane with vertices $(1,0)$, $(1,5)$, $(3,0)$, $(3,5)$?

Answer. Let R represent the rectangle, so that R is given by the inequalities $1 \leq x \leq 3, 0 \leq y \leq 5$. The volume then is given by $dV = z dA$, so is

$$V = \iint_R (\ln x + y) dA = \int_1^3 \left[\int_0^5 (\ln x + y) dy \right] dx .$$

The inner integral is

$$\int_0^5 (\ln x + y) dy = (y \ln x + \frac{y^2}{2}) \Big|_0^5 = 5 \ln x + 12.5 .$$

Thus

$$V = \int_1^3 (5 \ln x + 12.5) dx = 5(x \ln x - x) \Big|_1^3 + 12.5x \Big|_1^3 = 15 \ln 3 - 15 + 37.5 - 5 - 12.5 = 15 \ln 3 + 5 .$$

2. What is the volume of the solid bounded by the surfaces $z = x^3$ and $z = x^2 + 2y^2$ lying directly over the rectangle $0 \leq x \leq 1, 0 \leq y \leq 3$?

Answer. Note that for $0 \leq x \leq 1$ we must have $x^3 \leq x^2 \leq x^2 + 2y^2$. Thus we shall integrate over the rectangle $0 \leq x \leq 1, 0 \leq y \leq 3$, with the height ranging from x^3 to $x^2 + 2y^2$. Thus

$$V = \int_0^1 \left[\int_0^3 (x^2 + 2y^2 - x^3) dy \right] dx .$$

The inner integral is $3x^2 + 18 - 3x^3$, and the volume is 18.25 .

3. What is the volume of the solid bounded above by the surface $z = y^2 - x^2$ lying directly over the triangle $T : 0 \leq y \leq 2, -y \leq x \leq y$?

Answer. This is a type 2 regular domain, so

$$Volume = \iint_T z dA = \int_0^2 \left[\int_{-y}^y (y^2 - x^2) dx \right] dy .$$

The inner integral is

$$\int_{-y}^y (y^2 - x^2) dx = (y^2 x - \frac{x^3}{3}) \Big|_{-y}^y = \frac{4}{3} y^3 .$$

Then

$$Volume = \frac{4}{3} \int_0^2 y^3 dy = \frac{16}{3} .$$

4. What is the volume of the region under the surface $z = e^{x+y}$ and over the triangle in the xy -plane with vertices $(0,0)$, $(1,0)$, $(0,2)$.

Answer. This is the volume under the graph of $z = e^{x+y}$ and over a triangle T . T is bounded by the lines $x = 0, y = 0$ and $y = 2 - 2x$. Thus

$$V = \iint_T z dx dy = \int_0^1 \left[\int_0^{2-2x} e^{x+y} dy \right] dx .$$

The inner integral is

$$\int_0^{2-2x} e^{x+y} dy = e^{x+y} \Big|_0^{2-2x} = e^{2-x} - e^x .$$

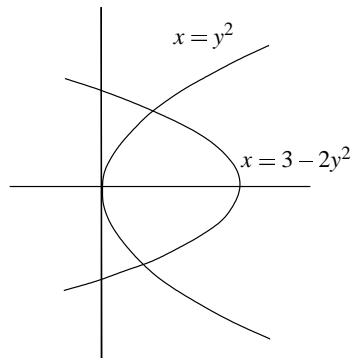
Now we integrate with respect to x :

$$\int_0^1 (e^{2-x} - e^x) dx = (-e^{2-x} - e^x) \Big|_0^1 = e^2 - 2e + 1 .$$

5. Let R be the region in the plane bounded by the curves $x = y^2$, $x = 3 - 2y^2$. Calculate

$$I = \iint_R (y^2 - x) dx dy .$$

Answer. Draw the region described by the curves:



Integrate first with respect to x along curves $y = \text{constant}$ from y^2 to $3 - 2y^2$. To find the range in y find the point of intersection of the parabolas: $y^2 = 3 - 2y^2$; the solutions are $y = \pm 1$. Thus

$$I = \int_{-1}^1 \left[\int_{y^2}^{3-2y^2} (y^2 - x) dx \right] dy .$$

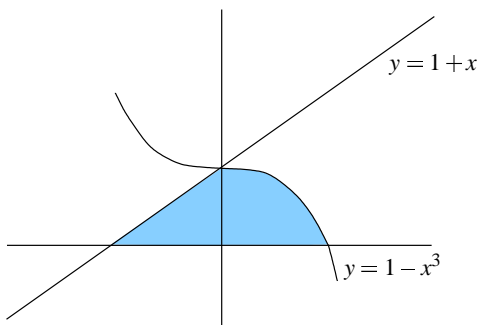
The inner integral is

$$\left[y^2 x - \frac{x^2}{2} \right] \Big|_{y^2}^{3-2y^2} = -2y^4 + 6y^2 - \frac{9}{2} .$$

Integrating over y , we obtain $I = -29/5$.

6. What is the mass of the lamina bounded by the curves $y = 1 + x$ and $y = 1 - x^3$ and the x -axis, where the density function is $\delta(x, y) = x^2$?

Answer. Draw the region described by the curves:



If we insist on viewing this as a type 1 domain, we must split the region into two parts: $-1 \leq x \leq 0$, $0 \leq y \leq 1 + x$; $0 \leq x \leq 1$, $0 \leq y \leq 1 - x^3$, and integrate $dydx$. As a type 2 domain, it is easier: the region is given as $0 \leq y \leq 1$, $y - 1 \leq x \leq (1 - y)^{1/3}$ and integrate $dx dy$. In the second case, the mass is given by

$$Mass = \int_0^1 \left[\int_{y-1}^{(1-y)^{1/3}} x^2 dx \right] dy .$$

The integral with respect to x is

$$\frac{1-y}{3} - \frac{(y-1)^3}{3} .$$

Now, to complete the integration, make the change of variable $u = 1 - y$:

$$\frac{1}{3} \int_0^1 ((1-y) - (y-1)^3) dy = \frac{1}{3} \int_0^1 (u + u^3) du = \frac{1}{4} .$$

7. A lamina filled with a homogeneous material (the density is identically equal to 1) is in the shape of the region R bounded by the curves $y = 1$ and $y = x^2$. What is its center of mass?

Answer. This is the region $-1 \leq x \leq 1$, $x^2 \leq y \leq 1$. Call the center of mass (\bar{x}, \bar{y}) . Because of symmetry, $\bar{x} = 0$. Now

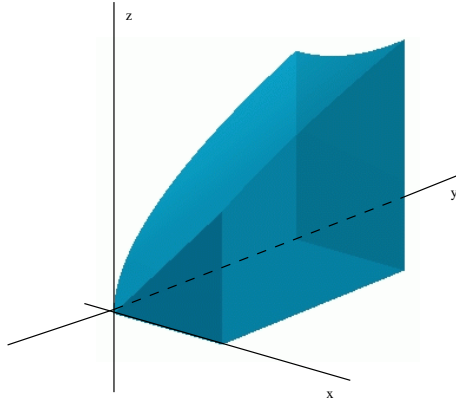
$$Mass = \int_{-1}^1 \int_{x^2}^1 dy dx = \frac{4}{3} ,$$

$$Mom_{y=0} = \int_{-1}^1 \int_{x^2}^1 y dy dx = \frac{4}{5} .$$

Thus $(\bar{x}, \bar{y}) = (0, 3/5)$.

8. Find the mass of the solid bounded by the surface $z = \sqrt{x^2 + y}$, the coordinate planes and the planes $x = 1$, $y = 2$, where the density is $\delta(x, y, z) = x$.

Answer. Draw the region bounded by the surfaces:



Now, over an infinitesimal rectangle of side lengths dx, dy , the mass is $dM = \delta dV = \delta z dx dy$, so

$$Mass = \int \int_R z \delta dx dy = \int_0^1 \int_0^2 x \sqrt{x^2 + y} dy dx .$$

$$\int_0^2 \sqrt{x^2 + y} dy = \frac{2}{3} (x^2 + y)^{3/2} \Big|_0^2 = \frac{2}{3} ((x^2 + 2)^{3/2} - x^3) .$$

$$Mass = \frac{2}{3} \int_0^1 x ((x^2 + 2)^{3/2} - x^3) dx = \frac{2}{3} \left(\frac{1}{5} (x^2 + 2)^{5/2} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{2}{15} [3^{5/2} - 1] .$$