Calculus III
Practice Problems 7: Answers

1. What is the volume of the region under the surface $z = \ln x + y$ and over the rectangle in the xy-plane with vertices (1,0), (1,5), (3,0), (3,5)?

Answer. Let R represent the rectangle, so that R is given by the inequalities $1 \le x \le 3, 0 \le y \le 5$. The volume then is given by $dV = zdA$, so is

$$V = \int\int_R (\ln x + y)dA = \int_1^3 \left[\int_0^5 (\ln x + y)dy\right]dx \ .$$

The inner integral is

$$\int_0^5 (\ln x + y)dy = (y\ln x + \frac{y^2}{2})_0^5 = 5\ln x + 12.5 \ .$$

Thus

$$V = \int_1^3 (5\ln x + 12.5)dx = 5(x\ln x - x)_1^3 + 12.5x|_1^3 = 15\ln 3 - 15 + 37.5 - 5 - 12.5 = 15\ln 3 + 5 \ .$$

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2. What is the volume of the solid bounded by the surfaces $z = x^3$ and $z = x^2 + 2y^2$ lying directly over the rectangle $0 \le x \le 1,\ 0 \le y \le 3$?
Answer. Note that for $0 \le x \le 1$ we must have $x^3 \le x^2 \le x^2 + 2y^2$. Thus we shall integrate over the rectangle $0 \le x \le 1,\ 0 \le y \le 3$, with the height ranging from x^3 to $x^2 + 2y^2$. Thus

$$V = \int_0^1 [\int_0^3 (x^2 + 2y^2 - x^3)dy]dx \ .$$

The inner integral is $3x^2 + 18 - 3x^3$, and the volume is 18.25 .

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3. What is the volume of the solid bounded above by the surface $z = y^2 - x^2$ lying directly over the triangle $T:\ 0 \le y \le 2, -y \le x \le y$?

Answer. This is a type 2 regular domain, so

$$Volume = \int\int_T zdA = \int_0^2 [\int_{-y}^y (y^2 - x^2)dx]dy \ .$$

The inner integral is

$$\int_{-y}^y (y^2 - x^2)dx = (y^2 x - \frac{x^3}{3})|_{-y}^y = \frac{4}{3}y^3 \ .$$

Then

$$Volume = \frac{4}{3}\int_0^2 y^3 dy = \frac{16}{3} \ .$$

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4. What is the volume of the region under the surface $z = e^{x+y}$ and over the triangle in the xy-plane with vertices (0,0), (1,0), (0,2).

Answer. This is the volume under the graph of $z = e^{x+y}$ and over a triangle T. T is bounded by the lines $x = 0$, $y = 0$ and $y = 2 - 2x$. Thus

$$V = \int\int_T zdxdy = \int_0^1 [\int_0^{2-2x} e^{x+y}dy]dx \ .$$

The inner integral is

$$\int_0^{2-2x} e^{x+y} dy = e^{x+y}\Big|_0^{2-2x} = e^{2-x} - e^x \ .$$

Now we integrate with respect to x:

$$\int_0^1 (e^{2-x} - e^x) dx = (-e^{2-x} - e^x)_0^1 = e^2 - 2e + 1 \ .$$

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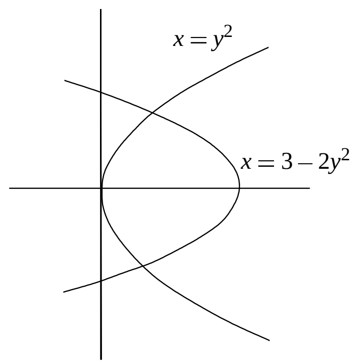
5. Let R be the region in the plane bounded by the curves $x = y^2$, $x = 3 - 2y^2$. Calculate

$$I = \int\int_R (y^2 - x) dx dy \ .$$

Answer. Draw the region described by the curves:



Integrate first with respect to x along curves y = constant from y^2 to $3 - 2y^2$. To find the range in y find the point of intersection of the parabolas: $y^2 = 3 - 2y^2$; the solutions are $y = \pm 1$. Thus

$$I = \int_{-1}^{1} \Big[\int_{y^2}^{3-y^2} (y^2 - x) dx \Big] dy \ .$$

The inner integral is

$$\Big[y^2 x - \frac{x^2}{2} \Big] \Big|_{y^2}^{3-y^2} = -2y^4 + 6y^2 - \frac{9}{2} \ .$$

Integrating over y, we obtain $I = -29/5$.

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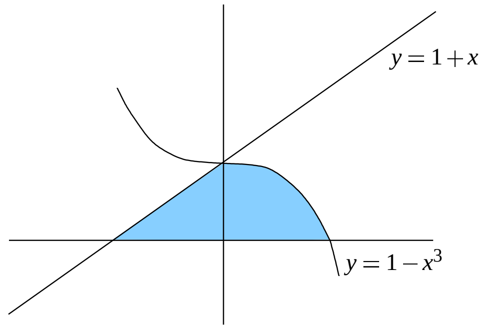
6. What is the mass of the lamina bounded by the curves $y = 1 + x$ and $y = 1 - x^3$ and the x-axis, where the density function is $\delta(x, y) = x^2$?

Answer. Draw the region described by the curves:



If we insist on viewing this as a type 1 domain, we must split the region into two parts: $-1 \leq x \leq 0,\ 0 \leq y \leq 1+x;\ \ 0 \leq x \leq 1,\ 0 \leq y \leq 1-x^3$, and integrate $dydx$. As a type 2 domain, it is easier: the region is given as $0 \leq y \leq 1,\ \ y-1 \leq x \leq (1-y)^{1/3}$ and integrate $dxdy$. In the second case, the mass is given by

$$Mass = \int_0^1 [\int_{y-1}^{(1-y)^{1/3}} x^2 dx] dy \ .$$

The integral with respect to dx is

$$\frac{1-y}{3} - \frac{(y-1)^3}{3} \ .$$

Now, to complete the integration, make the change of variable $u = 1-y$:

$$\frac{1}{3}\int_0^1 (((1-y)-(y-1)^3)dy = \frac{1}{3}\int_0^1 (u+u^3)du = \frac{1}{4} \ .$$

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7. A lamina filled with a homogeneous material (the density is identicaly equal to 1) is in the shape of the region R bounded by the curves $y = 1$ and $y = x^2$. What is its center of mass?

Answer. This is the region $-1 \leq x \leq 1,\ x^2 \leq y \leq 1$. Call the center of mass $(\overline{x}, \overline{y})$. Because of symmetry, $\overline{x} = 0$. Now

$$Mass = \int_{-1}^1 \int_{x^2}^1 dydx = \frac{4}{3} \ ,$$

$$Mom_{y=0} = \int_{-1}^1 \int_{x^2}^1 ydydx = \frac{4}{5} \ .$$

Thus $(\overline{x}, \overline{y}) = (0, 3/5)$.

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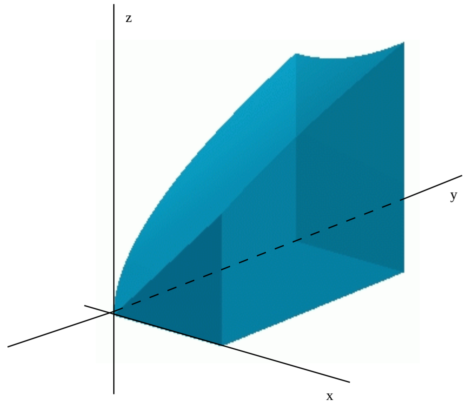
8. Find the mass of the solid bounded by the surface $z = \sqrt{x^2+y}$, the coordinate planes and the planes $x = 1$, $y = 2$, where the density is $\delta(x,y,z) = x$.

Answer. Draw the region bounded by the surfaces:



Now, over an infinitesimal rectangle of side lengths dx, dy, the mass is $dM = \delta dV = \delta z dx dy$, so

$$Mass = \int\int_R z\delta dx dy = \int_0^1 \int_0^2 x\sqrt{x^2+y}\, dy dx \ .$$

$$\int_0^2 \sqrt{x^2+y}\, dy = \frac{2}{3}(x^2+y)^{3/2}\Big|_0^2 = \frac{2}{3}((x^2+2)^{3/2} - x^3) \ .$$

$$Mass = \frac{2}{3}\int_0^1 x((x^2+2)^{3/2} - x^3)dx = \frac{2}{3}\left(\frac{1}{5}(x^2+2)^{3/2} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{2}{15}[3^{5/2} - 1] \ .$$