Calculus III
Practice Problems 6: Answers

1. Find the equation of the tangent plane to the surface given parametrically by

$$\mathbf{X}(u,v) = u^3\mathbf{I} + 2uv\mathbf{J} + v^2\mathbf{K}$$

at the point where $u = 1,\ v = 2$.

Answer. At the given values $u = 1,\ v = 2$, we have $\mathbf{X}(1,2) = \mathbf{I} + 4\mathbf{J} + 4\mathbf{K}$. The derivatives are

$$\mathbf{X_u} = 3u^2\mathbf{I} + 2v\mathbf{J}\ ; \quad \text{value at } (1,2) : 3\mathbf{I} + 4\mathbf{J}\ ,$$

$$\mathbf{X_v} = 2u\mathbf{J} + 2v\mathbf{K}\ ; \quad \text{value at } (1,2) : 2\mathbf{J} + 4\mathbf{K}\ .$$

We now calculate $\mathbf{X_u} \times \mathbf{X_v} = 16\mathbf{I} - 12\mathbf{J} + 6\mathbf{K}$, which we take to be the normal to the tangent plane. Thus the equation of the tangent plane, $\mathbf{N} \cdot \mathbf{X} = \mathbf{N} \cdot \mathbf{X_0}$ is $16x - 2y + 6z = -8$.

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2. Let S be a surface which goes through the origin, and whose normal is the z-axis. Let Π be a plane containing the z-axis, and γ the curve of intersection of the surface S and the plane Π. Show that the principal normal to γ is $\pm\mathbf{K}$.

Answer. Since the curve lies on the surface Σ, its tangent vector $\mathbf{T}$ is in the tangent plane to Σ, so is orthogonal to the z-axis. Thus $\mathbf{T} = a\mathbf{I} + b\mathbf{J}$ for some $a,\ b$. Since the curve lies in the plane Π, its acceleration vector lies in that plane, and thus its principal normal lies in that plane and is orthogonal to $\mathbf{T}$. The only unit vectors of that description are $\pm\mathbf{K}$.

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3. Let $f(x,y,z) = xyz - x^3 + x^2 + yz$. Find the critical points of f.
Answer. $\nabla f = (yz - 3x^2 + 2x)\mathbf{I} + (xz + z)\mathbf{J} + xy + y\mathbf{K}$. The equations for $\nabla f = 0$ are

$$yz - 3x^2 + 2x = 0\ , \quad xz + z = 0\ , \quad xy + y = 0\ .$$

From the second equation either $z = 0$ or $x = -1$.

Case $z = 0$: The first equation then tells us that either $x = 0$ or $x = 2/3$. In either case, the last equation tells us that $y = 0$. This gives us the two critical points (0,0,0) and (0,2/3,0).

Case $x = -1$. The first equation gives $yz = 5$, and the last equation 0=0 (no condition). Thus all points of the form $(-1, y, 5/y)$ with $y \neq 0$ are also critical points.

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4. Let

$$f(x,y) = x^3 - 4y^3 + 3x^2y - 18x + 6.$$

Find all critical points and classify as maxima, minima, saddle points.

Answer. First we calculate the first and second derivatives:

$$f_x = 3x^2 + 6xy - 18 = 3(x^2 + 2xy - 6)\ , \quad f_y = -12y^2 + 3x^2 = 3(x^2 - 4y^2)\ ,$$

$$f_{xx} = 6x + 6y\ , \quad f_{xy} = 6x\ , \quad f_{yy} = -24y\ ,$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 6(x+y)(-24y) - 36x^2 = -36(x^2 + xy + y^2)\ .$$

Now the equation $f_y = 0$ tells us that $x = \pm 2y$; substituting that in the formula for D, we have, that at any critical point (x,y) : $D = -36(4y^2 \pm 2y^2 + y^2) = -36y^2(5 \pm 1)$ is always negative. Thus all critical points are saddle points.

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5. Let

$$g(x,y,z) = x^2y^2z.$$

Find the point on the surface $g(x,y,z) = 1$ which is closest to the origin.

Answer. We want to minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $x^2y^2z = 1$. The gradients are

$$\nabla f = 2x\mathbf{I} + 2y\mathbf{J} + 2z\mathbf{K} , \quad \nabla g = 2xy^2z\mathbf{I} + 2x^2yz\mathbf{J} + x^2y^2\mathbf{K} ,$$

so the equations to solve are

$$(a) \quad xy^2z = \lambda x , \quad (b) \quad x^2yz = \lambda y , \quad (c) \quad x^2y^2 = \lambda z , \quad (d) \quad x^2y^2z = 1 .$$

Note that $x = y = z = \lambda = 1$ is a solution of these equations, so an answer is (1,1,1). However, if we divide (a) by (b), we get $y^2 = x^2$. Then (c) and (d) give us $x^2z = \lambda$, $x^4 = \lambda z$ which leads to $x^2 = z^2$. (d) tells us that z must be positive, so the closest points are $(\pm 1, \pm 1, 1)$.

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6. (This is a Calculus I problem. You are to do it using the methods of Lagrange multipliers). John and Mary work part-time at the Widget factory, and are willing to work as much as 40 hours a week. John gets paid \$27/hour, and Mary gets \$45/hour. If John works x hours and Mary works y hours, they produce $3xy + (1/2)y^{\ 2}$ widgets. The company has allocated \$1600/week for compensation to John and Mary (together). How many hours should they each work in order to produce the maximum number of widgets?

Answer. Let W be the number of widgets produced when John works x hours and Mary works y hours. We want to maximize $W = 3xy + (1/2)y^2$ subject to the constraint $g(x,y) = 27x + 45y = 1600$. We calculate the gradients, obtaining the Lagrange equations

$$3y = 27\lambda , \quad 3x + y = 45\lambda , \quad 27x + 45y = 1600 .$$

The first two equations allow us to express the third in terms of λ, getting $x = 12\lambda$, $y = 9\lambda$. Here notice that $x = (4/3)y$, so that, no matter how much there is to spend on labor, John works 33% more than Mary. In our case, we solve for λ putting this information in the last equation: $(27)(12\lambda) + (45)(9\lambda) = 1600$, giving $\lambda = 2.195$, $x = 16.34$, $y = 19.75$.

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7. The material for the bottom of a box costs three times as much per square foot as the material for the sides and top. We wish to know the greatest volume such a box can have if the total maount of money available for material is \$12, and the material for the bottom costs \$0.60 per square foot. Find the system of equations which must be solved to get the answer.

Answer. We want to maximize $V = xyz$, subject to the constraint
$C = .6xy + .2xy + 2(.2yz + .2xz) = 12$, The gradients of these functions are

$$\nabla V = yz\mathbf{I} + xz\mathbf{J} + xy\mathbf{K} ,$$

$$\nabla C = (.8y + .4z)\mathbf{I} + (.8x + .4z)\mathbf{J} + (.4y + .4x)\mathbf{K} \ ,$$

leading to the four equations in four unknowns:

$$(1) \quad yz = \lambda(.8y + .4z) \ , \quad (2) \quad xz = \lambda(.8x + .4z) \ , \quad (3) \quad xy = \lambda(.4y + .4x) \ ,$$
$$(4) \quad .8xy + 2(.2yz + 2xz) = 12 \ .$$

To solve, first multiply all equations by 10 to get rid of the decimal points, and replace 10λ by λ. Multiply (1) by x and (2) by y, and set the right hand sides equal to each other:

$$8\lambda xy + 4\lambda yz = 8\lambda xy + 4\lambda xz \ ,$$

from which we get $yz = xz$. Since all dimensions must be positive, we conclude that $x = y$. Substituting in (3), we get $x^2 = 8\lambda x$, so now we know $x = y = 8\lambda$. Put this in equation (2), and solve for z, getting $z = 16\lambda$. Now we find λ, by putting this all in equation (4) and solve, getting $\lambda = \sqrt{5/64} = \sqrt{5}/8)$. Thus the values producing the box of maximum volume are

$$x = \sqrt{5} \quad y = \sqrt{5} \quad z = 2\sqrt{5} \ .$$

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8. Let $f(x,y) = x^2 + 2y^2 + 2x$. Find the minimum and maximum of f on the ball $x^2 + y^2 \leq 16$.
Answer. The points we have to check are those inside the ball where $\nabla f = 0$, and then look for the maximum and minimum on the sphere $g(x,y) = x^2 + y^2 = 16$.
$\nabla f = (2x+2)\mathbf{I} + 4y\mathbf{J}$, so $\nabla f = 0$ at the point (-1,0).

To check for critical points on the sphere, we use the method of Lagrange multipliers. Since $\nabla g = 2x\mathbf{I} + 2y\mathbf{J}$, this leads to the system of equations

$$2x + 2 = 2\lambda x \ , \quad 4y = 2\lambda y \ , \quad x^2 + y^2 = 16 \ .$$

The second equation gives us the two cases: $y = 0$, or $\lambda = 2$.

If $y = 0$, from the last equation $x = \pm 4$.

If $\lambda = 2$, the first equation gives $x = 1$, and then the last gives $y = \pm\sqrt{15}$.

We now check the value of f at these points

(x,y)	$f(x,y)$
$(-1,0)$	-1
$(4,0)$	24
$(-4,0)$	8
$(1,\pm\sqrt{15})$	33

Thus the maximum of f is -1, at (-1,0) , and the maximum is 33, at the points $(1 \pm \sqrt{15})$.