Calculus III
Practice Problems 5: Answers

1. Let \( f(x, y, z) = x \ln z + 2yz \). a) What is \( \nabla f \)? b) Show that
\[
\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} .
\]

Answer. Differentiate with respect to the variables:
\[
\nabla f = \ln z \mathbf{I} + 2z \mathbf{J} + \left( \frac{x}{z} + 2y \right) \mathbf{K} .
\]
Now differentiate the coefficient of \( \mathbf{I} \) with respect to \( z \), and the coefficient of \( \mathbf{K} \) with respect to \( x \) and compare;
\[
\frac{\partial f}{\partial z \partial x} = \frac{1}{z} , \quad \frac{\partial f}{\partial x \partial z} = \frac{1}{z} .
\]

2. Find the direction \( \mathbf{U} \) of maximal change for the function \( w = x^3 y^2 z + xyz^2 \) at the point \( (2, -1, 2) \). What is \( D_\mathbf{U} w \) at this point?

Answer. The direction of maximal change is that of the gradient. We calculate the gradient:
\[
\nabla w = (3x^2 y^2 z + yz^2) \mathbf{I} + (2x^3 yz + xz^2) \mathbf{J} + (x^3 y^2 + 2xyz) \mathbf{K} .
\]
Evaluating at the given point, we have \( \nabla w = 20 \mathbf{I} - 24 \mathbf{J} \). \( \mathbf{U} \) is the unit vector in this direction, so \( \mathbf{U} = (5 \mathbf{I} - 6 \mathbf{J})/\sqrt{61} \). At this point \( D_\mathbf{U} w = |\nabla w| = 4\sqrt{61} \).

3. Suppose that the function \( w = f(x, y) \) is differentiable at the point \( P \) in the plane. Let \( \mathbf{V} = \mathbf{I} + 2 \mathbf{J} \), \( \mathbf{W} = \mathbf{I} - \mathbf{J} \), and that \( D_\mathbf{V} w = 2 \), \( D_\mathbf{W} w = 3 \) at \( P \). What is \( \nabla w(P) \)?

Answer. Let \( \nabla w(P) = a \mathbf{I} + b \mathbf{J} \). Then \( \nabla w \cdot \mathbf{V} = D_\mathbf{V} w = 2 \), \( \nabla w \cdot \mathbf{W} = D_\mathbf{W} w = 3 \), giving the equations
\[
a + 2b = 2 , \quad a - b = 3 .
\]
The solutions are \( a = 8/3 \), \( b = -1/3 \), so
\[
\nabla w(P) = \frac{8}{3} \mathbf{I} - \frac{1}{3} \mathbf{J} .
\]

4. Suppose \( z = f(x, y) \) is differentiable at \( (1, 1) \), and suppose that
\[
\frac{d}{dt} f(1 + t, 1 + t^2) \big|_{t=0} = 3 , \quad \frac{d}{dt} f(1, 1 + t) \big|_{t=0} = 2 .
\]
What is \( \nabla f(1, 1) \)?

Answer. Let \( \nabla f(1, 1) = a \mathbf{I} + b \mathbf{J} \). We are told the derivatives of \( f \) on the curves
\[
\gamma_1 : \quad X_1(t) = (1 + t) \mathbf{I} + (1 + t^2) \mathbf{J} , \quad \gamma_2 : \quad X_2(t) = \mathbf{I} + (1 + t) \mathbf{J} .
\]
We have
\[
\frac{dX_1}{dt} = \mathbf{I} + 2t \mathbf{J} , \quad \frac{dX_2}{dt} = \mathbf{J} ,
\]
and computing the limits as \( t \to 0 \) produces the values of \( a \) and \( b \).
so, at \( t_0, dX_1/dt = I, dX_2/dt = J \) and

\[
\begin{align*}
on \ \gamma_1 : \quad 3 \frac{df}{dt} &= \nabla f \cdot \frac{dX_1}{dt} = (aI + bJ) \cdot I = a, \\
on \ \gamma_2 : \quad 2 \frac{df}{dt} &= \nabla f \cdot \frac{dX_2}{dt} = (aI + bJ) \cdot J = b,
\end{align*}
\]

so \( \nabla f(1,1) = 3I + 2J \).

5. A particle moves in space according to the equation

\[
X(t) = t^2I + (1 - t^2)J + (1 - t)K,
\]

For \( w = xy + yz^2 + xz^2 \), find \( dw/dt \) along the trajectory. What is \( dw/dt \) when \( t = 2? \)

**Answer.** We know that

\[
\frac{dw}{dt} = \nabla w \cdot \frac{dX}{dt}.
\]

Now,

\[
\frac{dX}{dt} = 2tI - 2tJ - K,
\]

\[
\nabla w = (y + z^2)I + (x + z^2)J + 2z(x + y)K.
\]

Calculating the dot product:

\[
\frac{dw}{dt} = \nabla w \cdot \frac{dX}{dt} = (y + z^2)(2t) - (x + z^2)(2t) - 2z(x + y).
\]

At \( t = 2 \), we have \( x = 4, y = -3, z = -1 \), so

\[
\frac{dw}{dt} = (-3 + 1)(4) - (4 + 1)(4) - 2(-1)(4 - 3) = -22.
\]

6. Let \( w = xyz \) and let \( \gamma \) be the helix given by

\[
X(t) = \cos tI + \sin tJ + tk.
\]

Find \( dw/dt \) at \( t = 2\pi/3 \).

**Answer.** \( \nabla w = yeI + xeJ + xyK \), and for the helix,

\[
\frac{dX}{dt} = -\sin tI + \cos tJ + K.
\]

At \( t = 2\pi/3 \), \( x = \cos(2\pi/3) = -1/2, y = \sin(2\pi/3) = \sqrt{3}/2, z = 2\pi/3, \) so

\[
\nabla w = \frac{\sqrt{3}}{2}\frac{2\pi}{3}I + (-\frac{1}{2})\frac{2\pi}{3}J + \frac{2\pi}{3}K = \frac{\pi}{\sqrt{3}}I - \frac{\pi}{3}J + \frac{2\pi}{3}K, \quad \text{and} \quad \frac{dX}{dt} = \frac{1}{2}I + \frac{\sqrt{3}}{2}J + K.
\]

This gives

\[
\frac{dw}{dt} = \nabla w \cdot \frac{dX}{dt} = \frac{\pi}{2\sqrt{3}} - \frac{\pi\sqrt{3}}{6} + \frac{2\pi}{3} = \frac{2\pi}{3}.
\]
7. Find the equation of the tangent plane to the surface 

\[ x^{1/2} + y^{1/2} - z^{1/2} = 0 \]

at the point (4,9,25).

**Answer.** Take the differential of the defining equation:

\[ \frac{1}{2}x^{-1/2} dx + \frac{1}{2}y^{-1/2} dy - \frac{1}{2}z^{-1/2} dz = 0. \]

Multiply by 2 and evaluate at (4,9,25):

\[ \frac{dx}{2} + \frac{dy}{3} - \frac{dz}{5} = 0, \]

and replace the differentials by the increments:

\[ \frac{x - 4}{2} + \frac{y - 9}{3} - \frac{z - 25}{5} = 0. \]

When simplified, this comes down to \(15x + 10y - 6z = 0\).

8. Find the equation of the tangent plane to the surface 

\[ x \ln z + 2yz = 0 \]

at the point \((-e^2, 1, e^2)\).

**Answer.** Taking the differential, we have

\[ \ln z dx + \frac{x}{z} dz + 2ydz + 2zdy = 0. \]

Evaluate at the given point to get \((0)dx - e^2(e^{-2})dz + 2dz + 2e^2dy = dz + 2e^2dy = 0\). Now replacing the differentials by the increments we obtain

\[(z - e^2) + 2e^2(y - 1) = 0 \quad \text{or} \quad 2e^2y + z = 3e^2.\]