Calculus III
Practice Problems 5: Answers

1. Let $f(x,y,z) = x\ln z + 2yz$. a) What is ∇f? b) Show that

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z} \; .$$

Answer. Differentiate with respect to the variables:

$$\nabla f = \ln z\mathbf{I} + 2z\mathbf{J} + (\frac{x}{z} + 2y)\mathbf{K} \; .$$

Now differentiate the coefficient of **I** with respect to z, and the coefficient of **K** with respect to x and compare;

$$\frac{\partial f}{\partial z \partial x} = \frac{1}{z} \; , \quad \frac{\partial f}{\partial x \partial z} = \frac{1}{z} \; .$$

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2. Find the direction **U** of maximal change for the function $w = x^3y^2z + xyz^2$ at the point (2,-1,2). What is $D_{\mathbf{U}}w$ at this point?

Answer. The direction of maximal change is that of the gradient. We calculate the gradient:

$$\nabla w = (3x^2y^2z + yz^2)\mathbf{I} + (2x^3yz + xz^2)\mathbf{J} + (x^3y^2 + 2xyz)\mathbf{K} \; .$$

Evaluating at the given point, we have $\nabla w = 20\mathbf{I} - 24\mathbf{J}$. **U** is the unit vector in this direction, so $\mathbf{U} = (5\mathbf{I} - 6\mathbf{J})/\sqrt{61}$. At this point $D_{\mathbf{U}}w = |\nabla w| = 4\sqrt{61}$.

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3. Suppose that the function $w = f(x,y)$ is differentiable at the point P in the plane. Let $\mathbf{V} = \mathbf{I} + 2\mathbf{J}$, $\mathbf{W} = \mathbf{I} - \mathbf{J}$, and that $D_{\mathbf{V}}w = 2$, $D_{\mathbf{W}}w = 3$ at P. What is $\nabla w(P)$?

Answer. Let $\nabla w(P) = a\mathbf{I} + b\mathbf{J}$. Then $\nabla w \cdot \mathbf{V} = D_{\mathbf{V}}w = 2$, $\nabla w \cdot \mathbf{W} = D_{\mathbf{W}}w = 3$, giving the equations

$$a + 2b = 2 \quad a - b = 3 \; .$$

The solutions are $a = 8/3$, $b = -1/3$, so

$$\nabla w(P) = \frac{8}{3}\mathbf{I} - \frac{1}{3}\mathbf{J} \; .$$

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4. Suppose $z = f(x,y)$ is differentiable at (1,1), and suppose that

$$\frac{d}{dt}f(1+t, 1+t^2)|_{t=0} = 3, \quad \frac{d}{dt}f(1, 1+t)|_{t=0} = 2 \; .$$

What is $\nabla f(1,1)$?

Answer. Let $\nabla f(1,1) = a\mathbf{I} + b\mathbf{J}$. We are told the derivatives of f on the curves

$$\gamma_1 : \; \mathbf{X_1}(t) = (1+t)\mathbf{I} + (1+t^2)\mathbf{J} \; , \qquad \gamma_2 : \; \mathbf{X_2}(t) = \mathbf{I} + (1+t)\mathbf{J} \; .$$

We have

$$\frac{d\mathbf{X_1}}{dt} = \mathbf{I} + 2t\mathbf{J} \; , \quad \frac{d\mathbf{X_2}}{dt} = \mathbf{J} \; ,$$

so, at $t0$, $d\mathbf{X_1}/dt = \mathbf{I}$, $d\mathbf{X_2}/dt = \mathbf{J}$ and

$$\text{on } \gamma_1 : \quad 3 = \frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{X_1}}{dt} = (a\mathbf{I} + b\mathbf{J}) \cdot \mathbf{I} = a \,,$$

$$\text{on } \gamma_2 : \quad 2 = \frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{X_2}}{dt} = (a\mathbf{I} + b\mathbf{J}) \cdot \mathbf{I} = b \,,$$

so $\nabla f(1,1) = 3\mathbf{I} + 2\mathbf{J}$.

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5. A particle moves in space according to the equation

$$\mathbf{X}(t) = t^2\mathbf{I} + (1-t^2)\mathbf{J} + (1-t)\mathbf{K} \,.$$

For $w = xy + yz^2 + xz^2$, find dw/dt along the trajectory. What is dw/dt when $t = 2$?

Answer. We know that

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{X}}{dt} \,.$$

Now,

$$\frac{d\mathbf{X}}{dt} = 2t\mathbf{I} - 2t\mathbf{J} - \mathbf{K} \,,$$

$$\nabla w = (y + z^2)\mathbf{I} + (x + z^2)\mathbf{J} + 2z(x+y)\mathbf{K} \,.$$

Calculating the dot product:

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{X}}{dt} = (y + z^2)(2t) - (x + z^2)(2t) - 2z(x+y) \,.$$

At $t = 2$, we have $x = 4$, $y = -3$, $z = -1$, so

$$\frac{dw}{dt} = (-3+1)(4) - (4+1)(4) - 2(-1)(4-3) = -22 \,.$$

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6. Let $w = xyz$ and let γ be the helix given by

$$\mathbf{X}(t) = \cos t\,\mathbf{I} + \sin t\,\mathbf{J} + t\mathbf{k} \,.$$

Find dw/dt at $t = 2\pi/3$.

Answer. $\nabla w = yz\mathbf{I} + xz\mathbf{J} + xy\mathbf{K}$, and for the helix,

$$\frac{d\mathbf{X}}{dt} = -\sin t\,\mathbf{I} + \cos t\,\mathbf{J} + \mathbf{K} \,.$$

At $t = 2\pi/3$, $x = \cos(2\pi/3) = -1/2$, $y = \sin(2\pi/3) = \sqrt{3}/2$, $z = 2\pi/3$, so

$$\nabla w = \frac{\sqrt{3}}{2}\frac{2\pi}{3}\mathbf{I} + (-\frac{1}{2})\frac{2\pi}{3}\mathbf{J} + \frac{2\pi}{3}\mathbf{K} = \frac{\pi}{\sqrt{3}}\mathbf{I} - \frac{\pi}{3}\mathbf{J} + \frac{2\pi}{3}\mathbf{K} \,, \quad \text{and} \quad \frac{d\mathbf{X}}{dt} = \frac{1}{2}\mathbf{I} + \frac{\sqrt{3}}{2}\mathbf{J} + \mathbf{K} \,.$$

This gives

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{X}}{dt} = \frac{\pi}{2\sqrt{3}} - \frac{\pi\sqrt{3}}{6} + \frac{2\pi}{3} = \frac{2\pi}{3} \,.$$

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7. Find the equation of the tangent plane to the surface

$$x^{1/2}+y^{1/2}-z^{1/2}=0$$

at the point (4,9,25).
Answer. Take the differential of the defining equation:

$$\frac{1}{2}x^{-1/2}dx+\frac{1}{2}y^{-1/2}dy-\frac{1}{2}z^{-1/2}dy\ .$$

Multiply by 2 and evaluate at (4,9,25):

$$\frac{dx}{2}+\frac{dy}{3}-\frac{dz}{5}=0\ ,$$

and replace the differentials by the increments:

$$\frac{x-4}{2}+\frac{y-9}{3}-\frac{z-25}{5}=0\ .$$

When simplified, this comes down to $15x+10y-6z=0$.

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8. Find the equation of the tangent plane to the surface

$$x\ln z+2yz=0$$

at the point $(-e^2,1,e^2)$.

Answer. Taking the differential, we have

$$\ln z dx+\frac{x}{z}dz+2ydz+2zdy=0\ .$$

Evaluate at the given point to get $(0)dx-e^2(e^{-2})dz+2dz+2e^2dy=dz+2e^2dy=0$. Now replacing the differentials by the increments we obtain

$$(z-e^2)+2e^2(y-1)=0 \quad \text{or} \quad 2e^2y+z=3e^2\ .$$