Calculus III
Practice Problems 4: Answers

1. Consider the line L in the plane given by the equation $2x+5y+10=0$. Find a base $\{\mathbf{U},\mathbf{V}\}$ with $\mathbf{U}$ parallel to L, and $\mathbf{V}$ counterclockwise to $\mathbf{U}$. Find the equation of the line in coordinates $\{u,v\}$ relative to the base $\{\mathbf{U},\mathbf{V}\}$.

Answer. The vector $2\mathbf{I}+5\mathbf{J}$ is orthogonal to the line, so $5\mathbf{I}-2\mathbf{J}$ is in the direction of the line. Thus we can take

$$\mathbf{U}=\frac{5\mathbf{I}-2\mathbf{J}}{\sqrt{29}}\ ,\quad \mathbf{V}=\frac{2\mathbf{I}+5\mathbf{J}}{\sqrt{29}}\ .$$

If $\mathbf{X}=x\mathbf{I}+y\mathbf{J}=u\mathbf{U}+v\mathbf{V}$, then

$$u\mathbf{U}+v\mathbf{V}=u(\frac{5\mathbf{I}-2\mathbf{J}}{\sqrt{29}})+v(\frac{2\mathbf{I}+5\mathbf{J}}{\sqrt{29}})=(\frac{5u+2v}{\sqrt{29}})\mathbf{I}+(\frac{-2u+5v}{\sqrt{29}})\mathbf{J}=x\mathbf{I}+y\mathbf{J}\ ,$$

giving us x and y in terms of u,v. Putting these expressions in the equation gives the u,v equation of the line as

$$2(\frac{5u+2v}{\sqrt{29}})+5(\frac{-2u+5v}{\sqrt{29}})+10=0\ ,\quad \text{or}\quad v=-\frac{10}{\sqrt{29}}.$$

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2. An ellipse has center at the point (2,1), and its major axis is the line $x+y=3$. Its major radius is 3 and its minor radius is 1. What is the equation of the ellipse?

Answer. Choose coordinates u,v as in the figure on the next page. In these coordinates the equation of the ellipse is

$$\frac{u^2}{9}+v^2=1\ . \tag{1}$$

Now a vector in the direction of the line $x+y=5$ is $\mathbf{I}-\mathbf{J}$, so the base

$$\mathbf{L}=\frac{\mathbf{I}-\mathbf{J}}{\sqrt{2}}\ ,\quad \mathbf{M}=\frac{\mathbf{I}+\mathbf{J}}{\sqrt{2}}$$

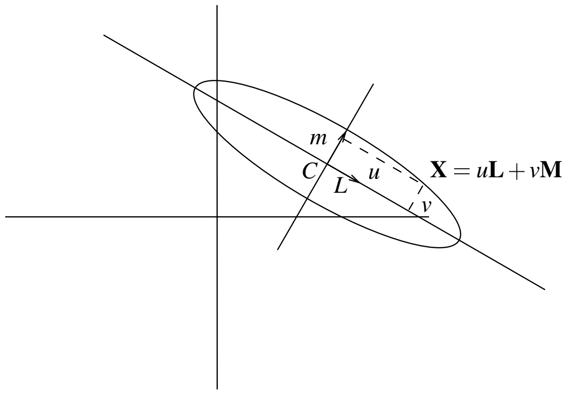
points in the directions of the axes of the ellipse. Now, the change of coordinates $\mathbf{X}=x\mathbf{I}+y\mathbf{J}=u\mathbf{L}+v\mathbf{M}$ is given by

$$u=(\mathbf{X}-(2\mathbf{I}+\mathbf{J}))\cdot\mathbf{L}=\frac{x-2-(y-1)}{\sqrt{2}}=\frac{x-y-1}{\sqrt{2}}\ ,$$
$$v=(\mathbf{X}-(2\mathbf{I}+\mathbf{J}))\cdot\mathbf{M}=\frac{x-2+(y-1)}{\sqrt{2}}=\frac{x+y-3}{\sqrt{2}}\ .$$

Substituting these into (1) gives us the equation

$$(x-y-1)^2+9(x+y-3)^2=18$$

which comes out to $10x^2+16xy+10y^2-56x-52y+64=0$.



3. Show that the intersection of a plane with a sphere is a circle.

Answer. Put the origin of the coordinates at the center of the sphere. Then the equation of the sphere is $|\mathbf{X}| = R$ for some $R > 0$. Let $\mathbf{N}$ be the unit normal to the plane, so that the equation of the plane is $\mathbf{X} \cdot \mathbf{N} = c$ for some c. Then, the intersection of the sphere and the plane is the circle in the plane with center $c\mathbf{N}$ and radius $\sqrt{R^2 - c^2}$. To check that, suppose that $\mathbf{X}$ is on the curve. Then

$$|\mathbf{X} - c\mathbf{N}|^2 = (\mathbf{X} - c\mathbf{N}) \cdot (\mathbf{X} - c\mathbf{N}) = \mathbf{X} \cdot \mathbf{X} - 2c\mathbf{X} \cdot \mathbf{N} + c^2\mathbf{N} \cdot \mathbf{N} = R^2 - c^2 ,$$

since $|\mathbf{X}| = R.$, $\mathbf{X} \cdot \mathbf{N} = c$ and $\mathbf{N}$ is of length one.

We could also argue this way: choose coordinates so that the sphere is centered at the origin, and the plane is perpendicular to the z axis. Then the equation of the sphere is $x^2 + y^2 + z^2 = R^2$, and the equation of the plane is $z = c$. These equations are equivalent to the equations $x^2 + y^2 = R^2 - c^2$, $z = c$, which are those of a circle on the plane $z = c$.

4. Consider the set of all points P in space such that the vector from O to P has length 2 and makes an angle of 45 degrees with $\mathbf{I} + \mathbf{J}$.

a) What kind of geometric object is this set?
b) Give equations in cartesian coordinates for this set.

Answer. If $\mathbf{X}$ is in this set, the conditions are $|\mathbf{X}| = 2$ and $\mathbf{X} \cdot (\mathbf{I} + \mathbf{J}) = |\mathbf{X}||\mathbf{I} + \mathbf{J}| \cos(45°)$. Writing $\mathbf{X} = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ gives us the equations

$$x^2 + y^2 + z^2 = 4 , \quad x + y = 2\sqrt{2}\frac{\sqrt{2}}{2} = 2 .$$

Thus the set is the intersection of the sphere of radius 2 with the plane $x + y = 2$: by problem 3, this is a circle.

5. Write down the equations of the paraboloid of revolution $z = x^2 + y^2$ in cylindrical and spherical coordinates.

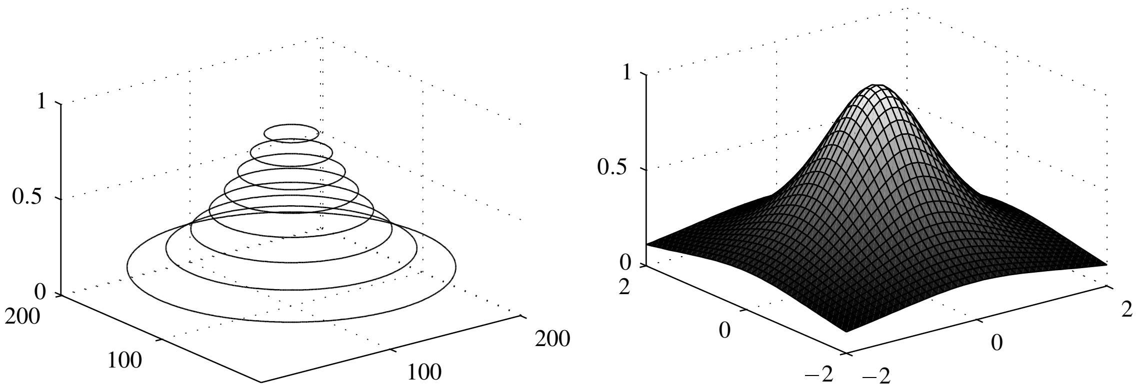
Answer. In cylindrical coordinates, $x^2 + y^2 = r^2$, so the equation is simply $z = r^2$. In spherical coordinates, this becomes $\rho \cos\phi = (\rho \sin\phi)^2$, or $\rho = \cot\phi \csc\phi$, for $0 < \phi \leq \pi/2$.

6. a) Draw some typical level curves in the (x,y)-plane for the function

$$f(x,y) = (1+x^2+y^2)^{-1}.$$

b) Sketch the surface $z = f(x,y)$.

Answer.



7. Let L be the line $x = 1$, $z = 3y$. If we rotate the line about the z-axis, it describes a surface. Find the equation of that surface.

Answer. For each z, the circle centered at $(0,0,z)$ and through the point $(1,z/3,z)$ is on the surface. The radius of this circle is $1+z^2/9$, thus we must have

$$x^2+y^2 = 1+\frac{z^2}{9} ,$$

which is therefore the equation of the surface.

8. Consider the helix $\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + t\mathbf{K}$. For each t, let L_t be the line perpendicular to the z-axis intersecting the helix at $\mathbf{X}(t)$. Find the equation of the surface swept out by the lines L_t. What are its level sets?

Answer. For any s, the point $\mathbf{X}(s,t) = s(\cos t\mathbf{I} + \sin t\mathbf{J}) + t\mathbf{K}$ is on L_t, so this is the parametric equation of the surface. To find the equation in cartesian coordinates, we must eliminate s,t for a relation between x,y,z. This is found by noting that

$$\frac{y}{x} = \frac{s\sin t}{s\cos t} = \tan t = \tan z$$

for all points $\mathbf{X}(s,t)$. Thus the equation of the surface is $y = x\tan z$.

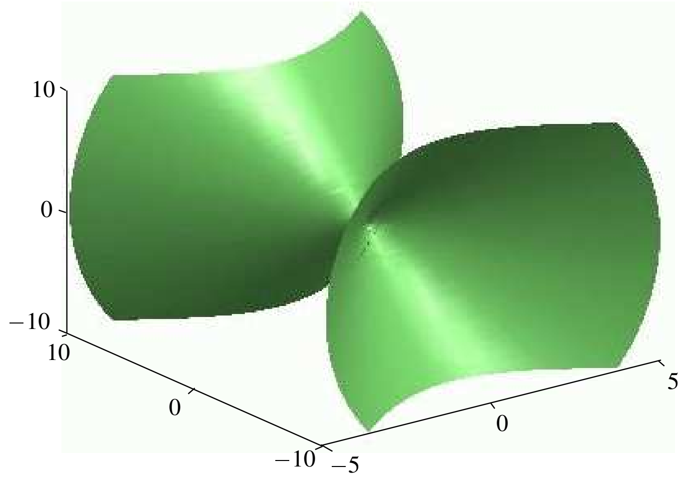
9. Sketch or describe the surface given by the equation

$$\frac{y^2}{4} - \frac{z^2}{9} = x^2.$$

Answer. Rewriting the equation as

$$x^2 - \frac{y^2}{4} + \frac{z^2}{9} = 0 \ ,$$

we see that the surface is a cone with axis the y-axis, and whose transverse sections are similar ellipses with major axis parallel to the z-axis and the ratio (major radius)/minor radius = 3/1. See the figure.



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10. Sketch or describe the surface given by the equation

$$\frac{x^2}{9} + \frac{z^2}{4} = y.$$

Answer. This is an elliptic paraboloid with axis the positive y-axis, and whose transverse sections are similar ellipses with major axis parallel to the x-axis and the ratio (major radius)/minor radius = 3/2. See figure the figure.