# Calculus III
## Practice Problems 3: Answers

1. Let $\mathbf{L} = \cos\theta(t)\mathbf{I} + \sin\theta(t)\mathbf{J}$ be a unit vector -valued function of $t$ in the $xy$-plane. Show that

$$\frac{d}{dt}(\mathbf{L}\times\mathbf{K}) = \frac{d\theta}{dt}\mathbf{L}.$$

**Answer**. $\mathbf{L}$ and $\mathbf{K}$ are unit vectors, so $\mathbf{L}\times\mathbf{K}$ is a unit vector orthogonal to $\mathbf{L}$ and $\mathbf{K}$ such that $\mathbf{L}$, $\mathbf{K}$, $\mathbf{L}\times\mathbf{K}$ is right-handed. Thus $\mathbf{L}\times\mathbf{K} = -\mathbf{L}^{\perp} = \sin\theta\mathbf{I} - \cos\theta\mathbf{J}$. Thus

$$\frac{d}{dt}(\mathbf{L}\times\mathbf{K}) = \cos\theta\frac{d\theta}{dt}\mathbf{I} + \sin\theta\frac{d\theta}{dt}\mathbf{J} = \frac{d\theta}{dt}\mathbf{L}.$$

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2. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = t\sin t\mathbf{I} + \cos t\mathbf{J}$$

Find the speed, tangential and normal accelerations and the curvature of the trajectory at the time $t = 2\pi$.

**Answer**. Differentiate;

$$\mathbf{V} = (\sin t + t\cos t)\mathbf{I} - \sin t\mathbf{J}\ , \quad \mathbf{A} = (\cos t + \cos t - t\sin t)\mathbf{I} - \cos t\mathbf{J}\ .$$

Evaluating at $t = 2\pi$, $\mathbf{V} = 2\pi\mathbf{I}$, $\mathbf{A} = 2\mathbf{I} - \mathbf{J}$. Then

$$\frac{ds}{dt} = 2\pi\ , \quad \mathbf{T} = \mathbf{I}\ , \quad \mathbf{N} = -\mathbf{J}$$

since $\mathbf{A}$ is clockwise from $\mathbf{T}$. This gives us $\mathbf{A} = \mathbf{2T} + \mathbf{N}$, so $a_T = a_N = 2$ and $\kappa = a_n/(ds/dt)^2 = 2/(4\pi^2)$.

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3. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = (t^2 + t + 1)\mathbf{I} + t^3\mathbf{J}$$

Find the velocity, speed, acceleration, tangent and normal vectors of the particle at any time $t > 0$.

**Answer**. Differentiate:

$$\mathbf{V} = (2t+1)\mathbf{I} + 3t^2\mathbf{J}\ , \quad \mathbf{A} = 2\mathbf{I} + 6t\mathbf{J}\ .$$

Then

$$\frac{ds}{dt} = \sqrt{(2t+1)^2 + 9t^4}\ , \quad \mathbf{T} = \frac{(2t+1)\mathbf{I} + 3t^2\mathbf{J}}{\sqrt{(2t+1)^2 + 9t^4}}\ , \quad \mathbf{N} = \frac{-3t^2\mathbf{I} + (2t+1)\mathbf{J}}{\sqrt{(2t+1)^2 + 9t^4}}\ .$$

$$a_T = \frac{2(2t+1) + (6t)(3t^2)}{\sqrt{(2t+1)^2 + 9t^4}}\ , \quad a_N = \frac{-6t^2 + (6t)(2t+1)}{\sqrt{(2t+1)^2 + 9t^4}} = \frac{6t^2 + 6t}{\sqrt{(2t+1)^2 + 9t^4}}\ .$$

Notice that the choice of sign for $\mathbf{N}$ was correct, since $a_N > 0$,

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4. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = t\mathbf{I} - \ln t\mathbf{J}$$

Find the velocity, speed, acceleration, tangent and normal vectors, and tangential and normal acceleration of the particle at any time $t > 0$.

**Answer**. Differentiate:

$$\mathbf{V} = \mathbf{I} - \frac{1}{t}\mathbf{J}\ , \quad \mathbf{A} = \frac{1}{t^2}\mathbf{J}\ .$$

$$\frac{ds}{dt} = \sqrt{1+\frac{1}{t^2}} = \frac{\sqrt{t^2+1}}{t}\ , \quad \mathbf{T} = \frac{t\mathbf{I}-\mathbf{J}}{\sqrt{t^2+1}}\ , \quad \mathbf{N} = \frac{\mathbf{I}+t\mathbf{J}}{\sqrt{t^2+1}}\ .$$

$$a_T = -\frac{1}{t^2\sqrt{t^2+1}}\ , \quad a_N = \frac{1}{t\sqrt{t^2+1}}$$

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5. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = e^{at}(\cos t\mathbf{I} + \sin t\mathbf{J})$$

Show that the angle between the position vector and the tangent line to the trajectory is constant.

**Answer**. Let that angle be $\beta$. We have

$$\mathbf{V} = ae^{at}(\cos t\mathbf{I} + \sin t\mathbf{J}) + e^{at}(-\sin t\mathbf{I} + \cos t\mathbf{J})\ .$$

We calculate:

$$\cos\beta = \frac{\mathbf{V}\cdot\mathbf{R}}{|\mathbf{V}||\mathbf{R}|} = \frac{ae^{2at}}{(e^{at}\sqrt{a^2+1})(e^{at})} = \frac{a}{\sqrt{a^2+1}}\ ,$$

so $\beta$ is constant. The easiest way to do this calculation is to introduce $\mathbf{L} = \cos t\mathbf{I} + \sin t + \mathbf{J}$, so that$\mathbf{R} = e^{at}\mathbf{L}$, $\mathbf{V} = e^{at}(a\mathbf{L} + \mathbf{L}^{\perp})$. Then, since $\mathbf{L}$ is a unit vector, we have

$$\mathbf{V}\cdot\mathbf{R} = ae^{2at}\ , \quad |\mathbf{R}| = e^{at}\ , \quad |\mathbf{V}| = e^{at}\sqrt{a^2+1}\ .$$

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6. A particle moves in space according to the formula

$$\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + \cos(2t)\mathbf{K}\ .$$

Find the tangential and normal accelerations and the curvature at $t = \pi/4$.

**Answer**. Differentiating:

$$\mathbf{V} = -\sin t\mathbf{I} + \cos t\mathbf{J} - 2\sin(2t)\mathbf{K}\ , \quad \mathbf{A} = -\cos t\mathbf{I} - \sin t\mathbf{J} - 4\cos(2t)\mathbf{K}\ .$$

At $t = \pi/4$:

$$\mathbf{V} = -\frac{1}{\sqrt{2}}\mathbf{I} + \frac{1}{\sqrt{2}}\mathbf{J} - 2\mathbf{K}\ , \quad \mathbf{A} = -\frac{1}{\sqrt{2}}\mathbf{I} - \frac{1}{\sqrt{2}}\mathbf{J}\ .$$

We have $ds/dt = \sqrt{5}$. Noting that $\mathbf{A}$ is orthogonal to $\mathbf{V}$, we conclude that $\mathbf{A} = a_N\mathbf{N}$, so that $a_T = 0$ and $a_N = |\mathbf{A}| = 1$, and finally, $\kappa = a_N/(ds/dt)^2 = 1/5$.

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7. A particle moves in space according to the formula

$$\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + e^t\mathbf{K} .$$

Find the tangential and normal accelerations for this motion.

**Answer**. Differentiate:

$$\mathbf{V} = -\sin t\mathbf{I} + \cos t\mathbf{J} + e^t\mathbf{K} , \quad \mathbf{A}(t) = -\cos t\mathbf{I} - \sin t\mathbf{J} + e^t\mathbf{K} .$$

Then $ds/dt = \sqrt{1+e^{2t}}$ and

$$a_T = \mathbf{A}\cdot\mathbf{T} = \frac{e^{2t}}{\sqrt{1+e^{2t}}} .$$

To find $a_N$ it is probably easiest to use the formula $a_N = |\mathbf{A}\times\mathbf{V}|/|\mathbf{V}|$. This calculation gives

$$a_N = \sqrt{\frac{1+2e^{2t}}{1+e^{2t}}} .$$

We can see this with less calculation by introducing $\mathbf{L} = \cos t\mathbf{I} + \sin t\mathbf{J}$, so that

$$\mathbf{X} = \mathbf{L} + e^t\mathbf{K} , \quad \mathbf{V} = \mathbf{L}^\perp + e^t\mathbf{K} , \quad \mathbf{A} = -\mathbf{L} + e^t\mathbf{K} .$$

Then $\mathbf{V}\times\mathbf{A}$ is easily computed since the system $\{\mathbf{L},\ \mathbf{L}^\perp, \mathbf{K}\}$ is a right handed system of orhogonal unit vectors. We get $\mathbf{V}\times\mathbf{A} = e^t\mathbf{L} - e^t\mathbf{L}^\perp + \mathbf{K}$, so $|\mathbf{V}\times\mathbf{A}|^2 = 1 + 2e^{2t}$.

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8. A particle moves in space according to the formula

$$\mathbf{X}(t) = \frac{1}{2}t^2\mathbf{I} + \frac{1}{t}\mathbf{J} - t\mathbf{K} .$$

Find $a_N, \kappa$ at the point $t = 1$.

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**Answer**. Differentiate

$$\mathbf{V} = t\mathbf{I} - \frac{1}{t^2}\mathbf{J} - \mathbf{K} , \quad \mathbf{A} = \mathbf{I} + \frac{2}{t^3}\mathbf{J} .$$

At $t = 1$, we obtain $\mathbf{V} = \mathbf{I} - \mathbf{J} + \mathbf{K}$, $\mathbf{A} = \mathbf{I} + 2\mathbf{J}$. Then

$$\frac{ds}{dt} = \sqrt{3} , \quad \mathbf{T} = \frac{\mathbf{I}-\mathbf{J}+\mathbf{K}}{\sqrt{3}} , \quad a_T = \frac{-1}{\sqrt{3}} .$$

We now calculate

$$a_N\mathbf{N} = \mathbf{A} - a_T\mathbf{T} = \mathbf{I} + 2\mathbf{J} - \frac{-\mathbf{I}+\mathbf{J}-\mathbf{K}}{3} = \frac{4\mathbf{I}+5\mathbf{J}+\mathbf{K}}{3} .$$

Then, $a_N$ is the magnitude of this vector:$a_N = \sqrt{42}/3$ and $\kappa = \sqrt{42}/9$ .

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9. Let $\mathbf{U}(t)$, $\mathbf{V}(t)$ be unit vectors which are always orthogonal. Let $\mathbf{W} = \mathbf{U}\times\mathbf{V}$. Show that there are scalar functions $\alpha$, $\beta$, $\gamma$ such that

$$\frac{d\mathbf{U}}{dt} = \alpha\mathbf{V} + \beta\mathbf{W} , \quad \frac{d\mathbf{V}}{dt} = -\alpha\mathbf{U} + \gamma\mathbf{W} , \quad \frac{d\mathbf{W}}{dt} = -\beta\mathbf{U} - \gamma\mathbf{V}$$

**Answer**. Since $\mathbf{U}, \mathbf{V}$ are orthogonal unit vectors, $\mathbf{W}$ is a unit vector orthogonal to both $\mathbf{U}$ and $\mathbf{V}$. Now, since $d\mathbf{U}/dt$ is orthogonal to $\mathbf{U}$, it lies in the plane of $\mathbf{V}$ and $\mathbf{W}$. Similarly, $d\mathbf{V}/dt$ lies in the plane of $\mathbf{U}$ and $\mathbf{W}$, and $d\mathbf{W}/dt$ lies in the plane of $\mathbf{U}$ and $\mathbf{V}$. This tells us that there are scalar functions $\alpha,\ \beta,\ \gamma, \delta, \mu,\ \nu$ such that

$$\frac{d\mathbf{U}}{dt} = \alpha\mathbf{V} + \beta\mathbf{W}\ , \quad \frac{d\mathbf{V}}{dt} = \delta\mathbf{U} + \gamma\mathbf{W}\ , \quad \frac{d\mathbf{W}}{dt} = \mu\mathbf{V} + \nu\mathbf{V}\ .$$

But now, since $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, $\mathbf{U}\cdot\mathbf{V} = 0$ so

$$\frac{d\mathbf{U}}{dt}\cdot\mathbf{V} + \mathbf{U}\cdot\frac{d\mathbf{V}}{dt} = 0\ ,$$

or $\delta = -\alpha$. The other identities, $\mu = -\beta,\ \nu = -\gamma$ are derived in the same way.

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10. Let $\mathbf{X} = \mathbf{X}(t)$ represent the motion of a particle in space.
a) Show that if $|\mathbf{V}|$ is constant, then $\mathbf{A}$ is orthogonal to $\mathbf{T}$.

**Answer**. Let $|\mathbf{V}| = c$, so that $\mathbf{V} = c\mathbf{T}$. Then

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = c\frac{d\mathbf{T}}{dt} = c\kappa\frac{ds}{dt}\mathbf{N}\ ,$$

so, since $\mathbf{N}$ is orthogonal to $\mathbf{T}$, so is $\mathbf{A}$. Another way to see this is to note that the hypothesis tells us that speed is constant, so

$$a_T = \frac{d^2s}{dt^2} = 0\ ,$$

so $\mathbf{A} = a_N\mathbf{N}$, and the conclusion follows.

b) Show that if $\mathbf{A}$ is constant, the motion lies in a plane.

**Answer**. By the hypothesis, $d\mathbf{V}/dt = \mathbf{C}$, a constant vector. Integrating, we get

$$\mathbf{V} = t\mathbf{C} + \mathbf{V_0}\ .$$

Integrating again:

$$\mathbf{X}(t) = \frac{t^2}{2}\mathbf{C} + t\mathbf{V_0} + \mathbf{X_0}\ ,$$

so that $\mathbf{X}(t) - \mathbf{X_0}$ is always in the plane spanned by $\mathbf{C}$ and $\mathbf{V_0}$.