1. Let \( \mathbf{L} = \cos \theta(t) \mathbf{I} + \sin \theta(t) \mathbf{J} \) be a unit vector-valued function of \( t \) in the \( xy \)-plane. Show that
\[
\frac{d}{dt}(\mathbf{L} \times \mathbf{K}) = \frac{d\theta}{dt} \mathbf{L}.
\]

**Answer.** \( \mathbf{L} \) and \( \mathbf{K} \) are unit vectors, so \( \mathbf{L} \times \mathbf{K} \) is a unit vector orthogonal to \( \mathbf{L} \) and \( \mathbf{K} \) such that \( \mathbf{L} \times \mathbf{K} = \mathbf{L} \times \mathbf{K} \) is right-handed. Thus \( \mathbf{L} \times \mathbf{K} = -\mathbf{L} \times \mathbf{K} \). Thus
\[
\frac{d}{dt}(\mathbf{L} \times \mathbf{K}) = \cos \theta \frac{d\theta}{dt} \mathbf{I} + \sin \theta \frac{d\theta}{dt} \mathbf{J} = \frac{d\theta}{dt} \mathbf{L}.
\]

2. A particle moves in the plane according to the equation
\[
\mathbf{X}(t) = t \sin t \mathbf{I} + \cos t \mathbf{J}
\]
Find the speed, tangential and normal accelerations and the curvature of the trajectory at the time \( t = 2\pi \).

**Answer.** Differentiate;
\[
\mathbf{V} = (\sin t + \cos t) \mathbf{I} - \sin t \mathbf{J}, \quad \mathbf{A} = (\cos t - \cos t \sin t) \mathbf{I} - \cos t \mathbf{J}.
\]
Evaluating at \( t = 2\pi \), \( \mathbf{V} = 2\pi \mathbf{I}, \mathbf{A} = 2\mathbf{I} - \mathbf{J} \). Then
\[
\frac{ds}{dt} = 2\pi, \quad \mathbf{T} = \mathbf{I}, \quad \mathbf{N} = -\mathbf{J}
\]
since \( \mathbf{A} \) is clockwise from \( \mathbf{T} \). This gives us \( \mathbf{A} = 2\mathbf{T} + \mathbf{N} \), so \( a_T = a_N = 2 \) and \( \kappa = a_n/(ds/dt)^2 = 2/(4\pi^2) \).

3. A particle moves in the plane according to the equation
\[
\mathbf{X}(t) = (t^2 + t + 1) \mathbf{I} + t^3 \mathbf{J}
\]
Find the velocity, speed, acceleration, tangent and normal vectors of the particle at any time \( t > 0 \).

**Answer.** Differentiate:
\[
\mathbf{V} = (2t + 1) \mathbf{I} + 3t^2 \mathbf{J}, \quad \mathbf{A} = 2\mathbf{I} + 6t \mathbf{J}.
\]
Then
\[
\frac{ds}{dt} = \sqrt{(2t+1)^2 + 9t^4}, \quad \mathbf{T} = \frac{(2t+1) \mathbf{I} + 3t^2 \mathbf{J}}{\sqrt{(2t+1)^2 + 9t^4}}, \quad \mathbf{N} = \frac{-3t^2 \mathbf{I} + (2t+1) \mathbf{J}}{\sqrt{(2t+1)^2 + 9t^4}}.
\]
\[
a_T = \frac{2(2t+1) + (6t)(3t^2)}{\sqrt{(2t+1)^2 + 9t^4}}, \quad a_N = \frac{-6t^2 + (6t)(2t+1)}{\sqrt{(2t+1)^2 + 9t^4}} = \frac{6t^2 + 6t}{\sqrt{(2t+1)^2 + 9t^4}}.
\]
Notice that the choice of sign for \( \mathbf{N} \) was correct, since \( a_N > 0 \),
4. A particle moves in the plane according to the equation
\[ \mathbf{X}(t) = t\mathbf{I} - \ln t\mathbf{J} \]

Find the velocity, speed, acceleration, tangent and normal vectors, and tangential and normal acceleration of the particle at any time \( t > 0 \).

**Answer.** Differentiate:
\[ \mathbf{V} = \mathbf{I} - \frac{1}{t}\mathbf{J}, \quad \mathbf{A} = \frac{1}{t^2}\mathbf{J}. \]
\[ \frac{d\mathbf{s}}{dt} = \sqrt{1 + \frac{1}{t^2}} = \frac{\sqrt{t^2 + 1}}{t}, \quad \mathbf{T} = \frac{t\mathbf{I} - \mathbf{J}}{\sqrt{t^2 + 1}}, \quad \mathbf{N} = \frac{\mathbf{I} + t\mathbf{J}}{\sqrt{t^2 + 1}}. \]
\[ a_T = -\frac{1}{t^2\sqrt{t^2 + 1}}, \quad a_N = \frac{1}{t\sqrt{t^2 + 1}}. \]

5. A particle moves in the plane according to the equation
\[ \mathbf{X}(t) = e^{at}(\cos t\mathbf{I} + \sin t\mathbf{J}) \]

Show that the angle between the position vector and the tangent line to the trajectory is constant.

**Answer.** Let that angle be \( \beta \). We have
\[ \mathbf{V} = ae^{at}(\cos t\mathbf{I} + \sin t\mathbf{J}) + e^{at}(-\sin t\mathbf{I} + \cos t\mathbf{J}). \]

We calculate:
\[ \cos \beta = \frac{\mathbf{V} \cdot \mathbf{R}}{|\mathbf{V}||\mathbf{R}|} = \frac{ae^{2at}}{(e^{at}\sqrt{a^2 + 1})(e^{at})} = \frac{a}{\sqrt{a^2 + 1}}, \]
so \( \beta \) is constant. The easiest way to do this calculation is to introduce \( \mathbf{L} = \cos t\mathbf{I} + \sin t\mathbf{J} \), so that \( \mathbf{R} = e^{at}\mathbf{L}, \mathbf{V} = e^{at}(a\mathbf{L} + \mathbf{L}') \). Then, since \( \mathbf{L} \) is a unit vector, we have
\[ \mathbf{V} \cdot \mathbf{R} = ae^{2at}, \quad |\mathbf{R}| = e^{at}, \quad |\mathbf{V}| = e^{at}\sqrt{a^2 + 1}. \]

6. A particle moves in space according to the formula
\[ \mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + \cos(2t)\mathbf{K}. \]

Find the tangential and normal accelerations and the curvature at \( t = \pi/4 \).

**Answer.** Differentiating:
\[ \mathbf{V} = -\sin t\mathbf{I} + \cos t\mathbf{J} - 2\sin(2t)\mathbf{K}, \quad \mathbf{A} = -\cos t\mathbf{I} - \sin t\mathbf{J} - 4\cos(2t)\mathbf{K}. \]

At \( t = \pi/4 \):
\[ \mathbf{V} = -\frac{1}{\sqrt{2}}\mathbf{I} + \frac{1}{\sqrt{2}}\mathbf{J} - 2\mathbf{K}, \quad \mathbf{A} = -\frac{1}{\sqrt{2}}\mathbf{I} - \frac{1}{\sqrt{2}}\mathbf{J}. \]

We have \( \frac{d\mathbf{s}}{dt} = \sqrt{3} \). Noting that \( \mathbf{A} \) is orthogonal to \( \mathbf{V} \), we conclude that \( \mathbf{A} = a_N\mathbf{N} \), so that \( a_T = 0 \) and \( a_N = |\mathbf{A}| = 1 \), and finally, \( \kappa = a_N/(d\mathbf{s}/dt)^2 = 1/3 \).
7. A particle moves in space according to the formula

\[ \mathbf{X}(t) = \cos t \mathbf{I} + \sin t \mathbf{J} + e^t \mathbf{K}. \]

Find the tangential and normal accelerations for this motion.

**Answer.** Differentiate:

\[ \mathbf{V} = - \sin t \mathbf{I} + \cos t \mathbf{J} + e^t \mathbf{K}, \quad \mathbf{A}(t) = - \cos t \mathbf{I} - \sin t \mathbf{J} + e^t \mathbf{K}. \]

Then \( ds/dt = \sqrt{1 + e^{2t}} \) and

\[ a_T = \mathbf{A} \cdot \mathbf{T} = \frac{e^{2t}}{\sqrt{1 + e^{2t}}} \cdot \mathbf{T}. \]

To find \( a_N \) it is probably easiest to use the formula \( a_N = |\mathbf{A} \times \mathbf{V}|/|\mathbf{V}|. \) This calculation gives

\[ a_N = \sqrt{\frac{1 + 2e^{2t}}{1 + e^{2t}}}. \]

We can see this with less calculation by introducing \( \mathbf{L} = \cos t \mathbf{I} + \sin t \mathbf{J}, \) so that

\[ \mathbf{X} = \mathbf{L} + e^t \mathbf{K}, \quad \mathbf{V} = \mathbf{L}^\perp + e^t \mathbf{K}, \quad \mathbf{A} = -\mathbf{L} + e^t \mathbf{K}. \]

Then \( \mathbf{V} \times \mathbf{A} \) is easily computed since the system \( \{\mathbf{L}, \mathbf{L}^\perp, \mathbf{K}\} \) is a right handed system of orthogonal unit vectors. We get \( \mathbf{V} \times \mathbf{A} = e^t \mathbf{L} - e^t \mathbf{L}^\perp + \mathbf{K} \) so \( |\mathbf{V} \times \mathbf{A}|^2 = 1 + 2e^{2t}. \)

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8. A particle moves in space according to the formula

\[ \mathbf{X}(t) = \frac{1}{2} t^2 \mathbf{I} + \frac{1}{t} \mathbf{J} - t \mathbf{K}. \]

Find \( a_N, \kappa \) at the point \( t = 1. \)

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**Answer.** Differentiate

\[ \mathbf{V} = \mathbf{I} - \frac{1}{t^2} \mathbf{J} - \mathbf{K}, \quad \mathbf{A} = \mathbf{I} + \frac{2}{t^3} \mathbf{J}. \]

At \( t = 1, \) we obtain \( \mathbf{V} = \mathbf{I} - \mathbf{J} + \mathbf{K}, \) \( \mathbf{A} = \mathbf{I} + 2 \mathbf{J}. \) Then

\[ \frac{ds}{dt} = \sqrt{3}, \quad \mathbf{T} = \frac{\mathbf{I} - \mathbf{J} + \mathbf{K}}{\sqrt{3}}, \quad a_T = \frac{-1}{\sqrt{3}}. \]

We now calculate

\[ a_N \mathbf{N} = \mathbf{A} - a_T \mathbf{T} = \mathbf{I} + 2 \mathbf{J} - \frac{-\mathbf{I} + \mathbf{J} - \mathbf{K}}{3} = \frac{4 \mathbf{I} + 5 \mathbf{J} + \mathbf{K}}{3}. \]

Then, \( a_N \) is the magnitude of this vector: \( a_N = \sqrt{42}/3 \) and \( \kappa = \sqrt{42}/9. \)

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9. Let \( \mathbf{U}(t), \mathbf{V}(t) \) be unit vectors which are always orthogonal. Let \( \mathbf{W} = \mathbf{U} \times \mathbf{V}. \) Show that there are scalar functions \( \alpha, \beta, \gamma \) such that

\[ \frac{d\mathbf{U}}{dt} = \alpha \mathbf{V} + \beta \mathbf{W}, \quad \frac{d\mathbf{V}}{dt} = -\alpha \mathbf{U} + \gamma \mathbf{W}, \quad \frac{d\mathbf{W}}{dt} = -\beta \mathbf{U} - \gamma \mathbf{V}. \]
Answer. Since $\mathbf{U}, \mathbf{V}$ are orthogonal unit vectors, $\mathbf{W}$ is a unit vector orthogonal to both $\mathbf{U}$ and $\mathbf{V}$. Now, since $d\mathbf{U}/dt$ is orthogonal to $\mathbf{U}$, it lies in the plane of $\mathbf{V}$ and $\mathbf{W}$. Similarly, $d\mathbf{V}/dt$ lies in the plane of $\mathbf{U}$ and $\mathbf{W}$, and $d\mathbf{W}/dt$ lies in the plane of $\mathbf{U}$ and $\mathbf{V}$. This tells us that there are scalar functions $\alpha, \beta, \gamma, \delta, \mu, \nu$ such that

$$
\frac{d\mathbf{U}}{dt} = \alpha \mathbf{V} + \beta \mathbf{W}, \quad \frac{d\mathbf{V}}{dt} = \delta \mathbf{U} + \gamma \mathbf{W}, \quad \frac{d\mathbf{W}}{dt} = \mu \mathbf{V} + \nu \mathbf{V}.
$$

But now, since $\mathbf{U}$ and $\mathbf{V}$ are orthogonal, $\mathbf{U} \cdot \mathbf{V} = 0$ so

$$
\frac{d\mathbf{U}}{dt} \cdot \mathbf{V} + \mu \mathbf{V} = 0,
$$

or $\delta = -\alpha$. The other identities, $\mu = -\beta, \nu = -\gamma$ are derived in the same way.

10. Let $\mathbf{X} = \mathbf{X}(t)$ represent the motion of a particle in space.

a) Show that if $|\mathbf{V}|$ is constant, then $\mathbf{A}$ is orthogonal to $\mathbf{T}$.

Answer. Let $|\mathbf{V}| = c$, so that $\mathbf{V} = c\mathbf{T}$. Then

$$
\mathbf{A} = \frac{d\mathbf{V}}{dt} = c \frac{d\mathbf{T}}{dt} = c \kappa \frac{ds}{dt} \mathbf{N},
$$

so, since $\mathbf{N}$ is orthogonal to $\mathbf{T}$, so is $\mathbf{A}$. Another way to see this is to note that the hypothesis tells us that speed is constant, so

$$
a_r = \frac{d^2s}{dt^2} = 0,
$$

so $\mathbf{A} = a_N \mathbf{N}$, and the conclusion follows.

b) Show that if $\mathbf{A}$ is constant, the motion lies in a plane.

Answer. By the hypothesis, $d\mathbf{V}/dt = \mathbf{C}$, a constant vector. Integrating, we get

$$
\mathbf{V} = \mathbf{tC} + \mathbf{V}_0.
$$

Integrating again:

$$
\mathbf{X}(t) = \frac{t^2}{2} \mathbf{C} + t \mathbf{V}_0 + \mathbf{X}_0,
$$

so that $\mathbf{X}(t) - \mathbf{X}_0$ is always in the plane spanned by $\mathbf{C}$ and $\mathbf{V}_0$. 