**Calculus III**
**Practice Problems 2: Answers**

1. Find a unit vector orthogonal to the vectors $\mathbf{V} = 6\mathbf{I} - 7\mathbf{J} + \mathbf{K}$, $\mathbf{W} = -\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$. What is the volume of the parallelopiped determined by these three vectors?

**Answer**. First, calculate

$$\mathbf{V} \times \mathbf{W} = \det \begin{pmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 6 & -7 & 1 \\ -1 & 2 & -3 \end{pmatrix} = (21-2)\mathbf{I} + (-1+18)\mathbf{J} + (12-7)\mathbf{K}$$

Now, the unit vector in the direction of $\mathbf{V} \times \mathbf{W}$ has length $\sqrt{19^2 + 17^2 + 5^2} = 25.98$, so the answer is $\mathbf{U} = .731\mathbf{I} + .654\mathbf{I} + .192\mathbf{K}$. Since $\mathbf{U}$ is a unit vector perpendicular to the plane of $\mathbf{V}$ and $\mathbf{W}$, the volume is the area of the base, $|\mathbf{V} \times \mathbf{W}| = 25.98$.

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2. Find a vector which makes an angle of $30°$ with the plane determined by the vectors $\mathbf{V} = 6\mathbf{I} - 7\mathbf{J} + \mathbf{K}$, $\mathbf{W} = -\mathbf{I} + 2\mathbf{J} - 3\mathbf{K}$.

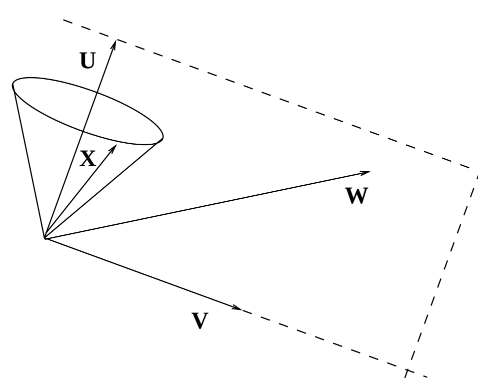
**Answer**. Since the vectors $\mathbf{V}$ and $\mathbf{W}$ are the same vectors as in the above problem, the unit normal to the plane of the vectors $\mathbf{V}$ and $\mathbf{W}$ is the vector $\mathbf{U}$ found there; $\mathbf{U} = .731\mathbf{I} + .654\mathbf{I} + .192\mathbf{K}$.The vector $\mathbf{X}$ we seek makes an angle of $60°$ with $\mathbf{U}$. There is such a vector in the plane determined by $\mathbf{U}$ and $\mathbf{V}$ (see the figure). So, if $\mathbf{M}$ is the unit vector in the direction of $\mathbf{V}$, we can take $\mathbf{X} = \cos 60°\mathbf{U} + \cos 30°\mathbf{M}$. First we calculate

$$\mathbf{M} = \frac{\mathbf{V}}{|\mathbf{V}|} = \frac{6\mathbf{I} - 7\mathbf{J} + \mathbf{K}}{\sqrt{36+49+1}} = .647\mathbf{I} - .755\mathbf{J} + .108\mathbf{K}\ .$$

Thus

$$\mathbf{X} = \cos 60°\mathbf{U} + \cos 30°\mathbf{M} = (.5)(.731\mathbf{I} + .654\mathbf{I} + .192\mathbf{K}) + (.866)(.647\mathbf{I} - .755\mathbf{J} + .108\mathbf{K})$$
$$= .926\mathbf{I} - .327\mathbf{J} + .190\mathbf{K}\ .$$

Of course there is a cone (as shown in the figure) of vectors solving this problem; we chose the unit vector in the plane of $\mathbf{V}$ and $\mathbf{V} \times \mathbf{W}$.

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3. Find a vector normal to the plane given by the equation

$$3x - 2y + z = 14\ .$$

**Answer**. $3\mathbf{I} - 2\mathbf{J} + \mathbf{K}$.

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4. Find a vector normal to the plane through $P : (0,0,0),\ Q : (1,0,-1),\ R : (0,1,1)$.
**Answer**. The vectors $\overrightarrow{PQ} = \mathbf{I} - \mathbf{K},\ \overrightarrow{PR} = \mathbf{J} + \mathbf{K}$ lie on the plane, so the vector we seek is

$$\mathbf{W} = \overrightarrow{PQ} \times \overrightarrow{PR} = (\mathbf{I} - \mathbf{K}) \times (\mathbf{J} + \mathbf{K}) = \mathbf{I} - \mathbf{J} + \mathbf{K}\ .$$

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5. Find the equation of the plane through the origin which is normal to the line given parametrically by

$$\mathbf{X} = (3\mathbf{I} + 2\mathbf{J} - \mathbf{K}) + t(-\mathbf{I} + \mathbf{J} + 2\mathbf{K})\ .$$

**Answer**. The vector $\mathbf{L} = -\mathbf{I} + \mathbf{J} + 2\mathbf{K}$ is parallel to the line, so can be taken as the normal to the desired plane. Thus the equation of the plane is $-x + y + 2z = 0$.

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6. Find $\mathbf{V_1} \cdot (\mathbf{V_2} \times \mathbf{V_3})$ where

$$\mathbf{V_1} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K}. \quad \mathbf{V_2} = 2\mathbf{I} - 2\mathbf{J} + 3\mathbf{K}, \quad \mathbf{V_3} = \mathbf{I} - 2\mathbf{K}\ .$$

**Answer**.

$$\mathbf{V_1} \cdot (\mathbf{V_2} \times \mathbf{V_3}) = \det \begin{pmatrix} \mathbf{V_1} \\ \mathbf{V_2} \\ \mathbf{V_3} \end{pmatrix} = \det \begin{pmatrix} -1 & 2 & 1 \\ 2 & -2 & 3 \\ 1 & 0 & -2 \end{pmatrix} = 12\ .$$

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7. Show that if $\det(\mathbf{U}, \mathbf{V}, \mathbf{W}) = |\mathbf{U}||\mathbf{V}||\mathbf{W}|$, then the three vectors are mutually orthogonal.

**Answer**. Let $\beta$ be the angle between $\mathbf{U}$ and the normal to the plane of $\mathbf{V}$ and $\mathbf{W}$, and $\alpha$ the angle between $\mathbf{V}$ and $\mathbf{W}$. Then

$$|\det(\mathbf{U}, \mathbf{V}, \mathbf{W})| = |\mathbf{U}||\mathbf{V} \times \mathbf{W}| \cos\beta = |\mathbf{U}||\mathbf{V}||\mathbf{W}| \cos\beta \sin\alpha\ .$$

If this is $|\mathbf{U}||\mathbf{V}||\mathbf{W}|$, then $\cos\beta \sin\alpha = 1$, so $\beta = 0$ and $\alpha = \pi/2$ or $\beta = \pi$ and $\alpha = -\pi/2$. In both cases the three vectors must be orthogonal.

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8. Find the symmetric equations of the line through the point (2,-1,3) which is perpendicular to the vectors $\mathbf{V} = 2\mathbf{I} - \mathbf{J} + 3\mathbf{K}$ and $\mathbf{W} = \mathbf{I} - \mathbf{J} + \mathbf{K}$.

**Answer**. The cross product $(2\mathbf{I} - \mathbf{J} + 3\mathbf{K}) \times (\mathbf{I} - \mathbf{J} + \mathbf{K}) = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$ has the direction of the line. Since $(2, -1, 3)$ is on the line, the symmetric equations are

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-1}\ .$$

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9. Find the parametric equations of the line through the point (2,-1,3) which is parallel to the two planes given by the equations

$$3x + z = 4, \qquad x - 2y + 5z = 1\ .$$

**Answer**. The line is parallel to the two planes, so is perpendicular to their normals. The normals are $\mathbf{N_1} = 3\mathbf{I} + 0\mathbf{J} + \mathbf{K},\ \mathbf{N_2} = \mathbf{I} - 2\mathbf{J} + 5\mathbf{K}$, so $\mathbf{N_1} \times \mathbf{N_2}$ is in the direction of the line. We find $\mathbf{N_1} \times \mathbf{N_2} = 2\mathbf{I} - 14\mathbf{J} - 6\mathbf{K}$, so the parametric equations of the line are:

$$x = 2 + 2t\ , \quad y = -1 - 14t\ , \quad z = 3 - 6t\ .$$

10. Find the equation of the plane through the point (2,-1,3) which is parallel to the vectors $\mathbf{I}-2\mathbf{J}+3\mathbf{K}$ and $3\mathbf{I}-2\mathbf{J}+\mathbf{K}$.

**Answer**. The normal to the plane is the cross product of the two given vectors. This is $\mathbf{N}=4\mathbf{I}+8\mathbf{J}+4\mathbf{K}$. Taking $\mathbf{X_0}=2\mathbf{I}-\mathbf{J}+3\mathbf{K}$, the equation is $(\mathbf{X}-\mathbf{X_0})\cdot\mathbf{N}=0$, which comes to $4x+8y+4z=12$.

11. Find the distance of the point $(2,0,1)$ from the plane given by the equation

$$\frac{x-2}{3}+\frac{y+1}{4}+\frac{z-1}{2}=0$$

**Answer**. Represent the given point by the vector $\mathbf{X}=2\mathbf{I}+\mathbf{K}$. The point $\mathbf{X_0}=2\mathbf{I}-\mathbf{J}+\mathbf{K}$ is on the given plane, and the normal to the plane is $\mathbf{N}=(1/3)\mathbf{I}+(1/4)\mathbf{J}+(1/2)\mathbf{K}$. The distance then is

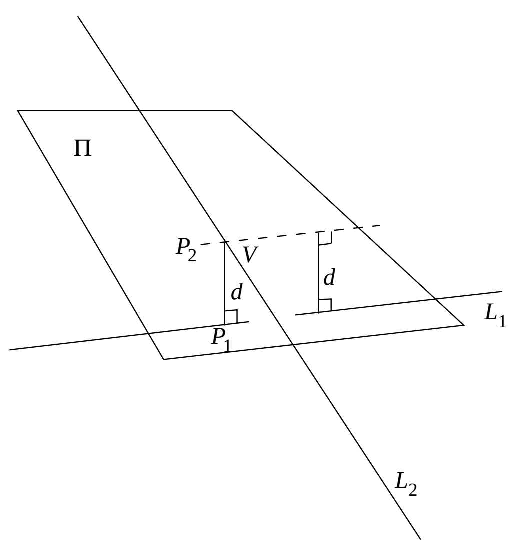
$$d=\frac{|(\mathbf{X}-\mathbf{X_0})\cdot\mathbf{N}|}{|\mathbf{N}|}=\frac{|\mathbf{J}\cdot((1/3)\mathbf{I}+(1/4)\mathbf{J}+(1/2)\mathbf{K})}{.651}=.384\ .$$

12. Let $L_1$, $L_2$ be two lines in three dimensions which do not intersect. There are points $P_1$, $P_2$ on $L_1$, $L_2$ respectively such that the line joining $P_1$ and $P_2$ intersects each line at a right angle. These are the points on the two lines which are closest together. Find a formula for the distance between $P_1$ and $P_2$, in terms of the equations of the lines

$$L_1: \quad \mathbf{X}=\mathbf{Q_1}+t\mathbf{N_1} \qquad L_2: \quad \mathbf{X}=\mathbf{Q_2}+s\mathbf{N_2}\ .$$

**Answer**. Let $\Pi$ be the plane containing the line $L_1$ and parallel to the line $L_2$. Then the distance between $P_1$ and $P_2$ is the same as the distance of any point on $L_1$ from the plane $\Pi$ (see the figure below). Now, the normal to $\Pi$ is $\mathbf{N_1}\times\mathbf{N_2}$, and $\mathbf{Q_1}$ is a point on $L_1$. Thus, using the distance formula, where we take $\mathbf{Q_2}$ as a point on $L_2$ we get

$$d=\frac{|(\mathbf{Q_2}-\mathbf{Q_1})\cdot(\mathbf{N_1}\times\mathbf{N_2})|}{|\mathbf{N_1}\times\mathbf{N_2}|}\ .$$




If the geometry isn't clear, we can argue as follows: Since the line joining $\mathbf{P_1}$ and $\mathbf{P_2}$ is perpendicular to $L_1$ and $L_2$, it has the direction $\mathbf{N_1}\times\mathbf{N_2}$. Now, the distance from $Q_2$ to $P_1$ is the projection of $\mathbf{Q_2}-\mathbf{P_1}$ in this direction, so is

$$d = \frac{|(\mathbf{Q_2} - \mathbf{P_1}) \cdot (\mathbf{N_1} \times \mathbf{N_2})|}{|\mathbf{N_1} \times \mathbf{N_2}|} \ .$$

But we don't know $P_1$, so we can't use this formula. However, since $\mathbf{Q_1}$ lies on the line $L_1$, we must have $Q_1 = \mathbf{P_1} + t\mathbf{N_1}$. Then, we can replace $\mathbf{P_1}$ by $\mathbf{Q_1}$, as this computation shows:

$$(\mathbf{Q_2} - \mathbf{Q_1}) \cdot (\mathbf{N_1} \times \mathbf{N_2}) = (\mathbf{Q_2} - \mathbf{P_1} + t\mathbf{N_1}) \cdot (\mathbf{N_1} \times \mathbf{N_2}) = (\mathbf{Q_2} - \mathbf{P_1}) \cdot (\mathbf{N_1} \times \mathbf{N_2}) + t\mathbf{N_1} \cdot (\mathbf{N_1} \times \mathbf{N_2}) = (\mathbf{Q_2} - \mathbf{P_1}) \cdot (\mathbf{N_1} \times \mathbf{N_2})$$

since $\mathbf{N_1}$ is perpendicular to $\mathbf{N_1} \times \mathbf{N_2}$.