Calculus III
Practice Problems 11: Answers

1. Let C be the boundary of the triangle with vertices at (0,0), (3,0), (4,5), oriented in the counterclockwise sense. Find $\int_C 3ydx + 6xdy$.

Answer. Green's theorem tells us this is

$$\int\int_T (6-3)dA = 3(Area(T)) = \frac{45}{2} ,$$

since T has base length 3 and altitude 5.

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2. Find $\int_C xy^2dx + x^2ydy$ where C is the line from (2,3) to (5,1).

Answer. Notice that the differential has a potential function:

$$xy^2dx + x^2ydy = d(\frac{x^2y^2}{2}) .$$

Thus

$$\int_C xy^2dx + x^2ydy = \frac{x^2y^2}{2}\Big|_{(2,3)}^{(5,1)} = -\frac{11}{2} .$$

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3. Let C be the boundary of the square $0 \leq x \leq \pi,\ 0 \leq x \leq \pi$, traversed in the counterclockwise sense. Find

$$\int_C \sin(x+y)dx + \cos(x+y)dy .$$

Answer. Let I be the integral to compute; by Green's theorem

$$I = \int_0^\pi \int_0^\pi (-\sin(x+y) - \cos(x+y))dydx .$$

The inner integral is

$$(\cos(x+y) - \sin(x+y))\Big|_0^\pi = -2\cos x + 2\sin x)$$

using the trig identities: $\cos(x+\pi) = -\cos x,\ \sin(x+\pi) = -\sin x$. Then

$$I = 2\int_0^\pi (\sin x - \cos x)dx = 4 .$$

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4. Let C be the boundary of the triangle with vertices (0,0), (1,0), (1,2), traversed in the counterclockwise sense. Find $\int_C x^2dx + xydy$.

Answer. The curve is the boundary of the triangle $D:\ 0 \leq x \leq 1,\ 0 \leq y \leq 2x$. Green's theorem gives us

$$\int_C x^2dx + xydy = \int_0^1 \int_0^{2x} ydydx = \int_0^1 \frac{(2x)^2}{2}dx = \frac{2}{3} .$$

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5. The **cycloid** is the curve given parametrically in the plane by

$$x = t - \sin t , \quad y = 1 - \cos t , \quad t \geq 0 .$$

Find the area under one arch of the cycloid.

Answer. The arch A is bounded below by $y=0,\ 0\le x\le 2\pi$, and above by the cycloid traversed in the direction opposite to the given parametrization. Using Green's theorem we have

$$Area = -\oint_{\partial A} y\,dx = \int_0^{2\pi}(1-\cos t)(1-\cos t)dt = \int_0^{2\pi}(1-2\cos t+\cos^2 t)dt = 2\pi+\pi = 3\pi\ .$$

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6. A function of two variables is **harmonic** if

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0\ .$$

Let R be a regular region in the plane with boundary C, $\mathbf{N}$ the exterior normal to C. Show that

$$\oint_C \nabla f\cdot\mathbf{N}ds = 0\ .$$

Answer. By the divergence theorem,

$$\oint_C \nabla f\cdot\mathbf{N}ds = \int\int \div\nabla f dA\ .$$

But

$$\div\nabla f = \div\left(\frac{\partial f}{\partial x}\mathbf{I}+\frac{\partial f}{\partial y}\mathbf{J}\right) = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0\ ,$$

since f is harmonic.

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7. Let $\mathbf{F}(\mathbf{X}) = e^{xy}(\mathbf{I}+\mathbf{J})$, and let R be the rectangle $0\le x\le 2,\ -1\le y\le 1$. For C the boundary of R traversed counterclockwise, Calculate

$$\text{a)}\quad \oint_C \mathbf{F}\cdot\mathbf{T}ds \qquad \text{b)}\quad \oint_C \mathbf{F}\cdot\mathbf{N}ds\ ,$$

where $\mathbf{T}$ and $\mathbf{N}$ are the tangent and normal to the curve C.

Answer. a). By Stokes' theorem:

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \oint_C e^{xy}(dx+dy) = \int\int_R (ye^{xy}-xe^{xy})dA$$

$$= \int_{-1}^1\int_0^2 ye^{xy}dxdy - \int_0^2\int_{-1}^1 xe^{xy}dydx\ .$$

The inner integrals are

$$\int_0^2 ye^{xy}dx = e^{2y}-1 \qquad \int_{-1}^1 xe^{xy}dy = e^x-e^{-x}\ ,$$

and

$$\int_{-1}^1 (e^{2y}-1)dy = \frac{e^2}{2}-\frac{e^{-2}}{2}-2\ , \qquad \int_0^2 (e^x-e^{-x})dx = e^2+e^{-2}-2\ ,$$

so

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = -\frac{1}{2}e^2-\frac{3}{2}e^{-2}\ .$$

b). By the divergence theorem, and finally, using the same computations as above, we get

$$\oint_C \mathbf{F}\cdot\mathbf{N}ds = \oint_C e^{xy}(-dx+dy) = \int\int_R (ye^{xy}+xe^{xy})dA = \frac{3}{2}e^2+\frac{1}{2}e^{-2}-4\ .$$

This may be a case where direct computation is easier than Green's theorem, because the boundary consists the four line segments $C_1 : y = -1,\ C_2 : x = 2,\ C_3 : y = 1,\ C_4 : x = 0.$ This gives us (taking into account the endpoints of the intervals and their orientations):

$$\oint_C \mathbf{F}\cdot\mathbf{T}ds = \oint_C e^{xy}(dx+dy) = \int_0^2 e^{-x}dx + \int_{-1}^1 e^{2y}dy + \int_2^0 e^x dx + \int_1^{-1} dy\ ,$$

four easier integrations to perform.

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8. Let $\mathbf{F} = x\mathbf{I} + xy^2\mathbf{J}$. Let C be the circle $x^2+y^2 = 1$ traversed in the counterclockwise sense. Find

$$\int_C \mathbf{F}\cdot\mathbf{T}ds\ , \qquad \int_C \mathbf{F}\cdot\mathbf{N}ds$$

where s represents arclength along the circle, $\mathbf{T}$ is the unit tangent vector, and $\mathbf{N}$ is the unit external normal vector.

Answer. Remember that, along any curve, where s denotes arc length, we have

$$d\mathbf{X} = dx\mathbf{I} + dy\mathbf{J} = \mathbf{T}ds\ ,$$

so, since $\mathbf{N}$ is orthogonal to and to the right of $\mathbf{T}$,

$$\mathbf{N}ds = dy\mathbf{I} - dx\mathbf{J}\ .$$

Using the parametrization of C: $\mathbf{X}(\theta) = \cos\theta\mathbf{I} + \sin\theta\mathbf{J},\ 0 \le \theta \le 2\pi$, we have

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C xdx + xy^2dy = \int_0^{2\pi}(-\cos\theta\sin\theta + \cos^2\theta\sin^2\theta)d\theta = \frac{1}{4}\int_0^{2\pi}\sin^2(2\theta)d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}(1-2\cos(4\theta))d\theta = \frac{\pi}{4}\ .$$

Similarly

$$\int_C \mathbf{F}\cdot\mathbf{N}ds = \int_C xdy - xy^2dx = \int_0^{2\pi}(\cos^2\theta + \cos\theta\sin^3\theta)d\theta = \pi\ .$$

Using Green's theorem the calculations are a little easier. Letting D denote the unit disc:

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int\int_D \mathrm{curl}\ \mathbf{F}\cdot\mathbf{K}dxdy = \int\int_D y^2dxdy = \int_0^{2\pi}\int_0^1 r^3\cos^2\theta drd\theta = \frac{\pi}{4}\ ,$$

$$\int_C \mathbf{F}\cdot\mathbf{N}ds = \int\int_D \mathrm{div}\ \mathbf{F}dA = \int\int_D(1+2xy)dA = \int_0^{2\pi}\int_0^1(1+r^2\cos\theta\sin\theta)rdrd\theta = \pi\ .$$

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9. Let $\mathbf{F} = y\mathbf{I} + 2x\mathbf{J}$. Let C be the curve $y = x^2$ from $x = 0$ to $x = 2$. Find the flux of $\mathbf{F}$ across C from left to right, that is, for $\mathbf{N}$ the unit normal to the right along C, find

$$\int_C \mathbf{F}\cdot\mathbf{N}ds\ .$$

Answer. C has the parametrization $\mathbf{X}(x) = x\mathbf{I} + x^2\mathbf{J},\ 0 \le x \le 2.$ Thus $d\mathbf{X} = (\mathbf{I} + 2x\mathbf{J})dx$, so $\mathbf{N}ds = (2x\mathbf{I} - \mathbf{J})dx$. Along the curve $\mathbf{F} = x^2\mathbf{I} + 2x\mathbf{J}$. We thus have

$$\int_C \mathbf{F}\cdot\mathbf{N}ds\ = \int_0^2((2x^3-2x)dx = \frac{x^4}{2} - x^2\Big|_0^2 = \frac{16}{2} - 4 = 4\ .$$

10. Let $\mathbf{F} = x^3\mathbf{I} + y^3\mathbf{J}$. Let C be the circle $x^2 + y^2 = 9$ traversed in the counterclockwise sense. Find

$$\int_C \mathbf{F}\cdot\mathbf{N}ds \; .$$

where s represents arclength along the circle, and $\mathbf{N}$ is the unit external normal vector.

Answer. Here , if you work directly, parametrize the circle by $\mathbf{X} = 3\cos\theta\mathbf{I} + 3\sin\theta\mathbf{J}$. Then $\mathbf{N}ds = dy\mathbf{I} - dx\mathbf{J} = 3(\cos\theta\mathbf{I} + \sin\theta\mathbf{J})d\theta$ and, since $x = 3\cos\theta$ and $y = 3\sin\theta$ along the curve:

$$\int_C \mathbf{F}\cdot\mathbf{N}ds \; = 3^4\int_0^{2\pi}(\cos^4\theta + \sin^4\theta)d\theta \; .$$

That's an ugly integral: use a table of integrals. Alternatively, use the double angle formulae several times to get

$$\cos^4\theta + \sin^4\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta \; .$$

Then,

$$\int_C \mathbf{F}\cdot\mathbf{N}ds \; = 3^4\int_0^{2\pi}(\frac{3}{4} + \frac{1}{4}\cos 4\theta)d\theta = 3^5\pi/2 \; .$$

On the other hand, the divergence theorem gives us

$$\int_C \mathbf{F}\cdot\mathbf{N}ds \; = \int\int_D \text{div}\;\mathbf{F}dA = \int\int_D (3x^2 + 3y^2)dA \; ,$$

where D is the disc $x^2 + y^2 \leq 9$. Changing to polar coordinates:

$$\int_C \mathbf{F}\cdot\mathbf{N}ds \; = \int_0^{2\pi}\int_0^3 (3r^2)rdrd\theta \; = 3(2\pi)(3^4)/4 == 3^5\pi/2 \; .$$