Calculus III
Practice Problems 10: Answers

1. Suppose that a fluid is rotating about the z-axis with constant angular speed ω. Let $\mathbf{V}$ be the velocity field of the motion.

a) Show that $\mathbf{V} = \omega\mathbf{K}\times\mathbf{X}$.

Answer. The particles move along the circles $x^2+y^2=a^2$, $z=b$ at speed ωa. Thus the velocity vector at a point $\mathbf{X} = x\mathbf{I}+y\mathbf{J}+z\mathbf{K}$ is in the direction of the tangent to this circle and of magnitude $\omega\sqrt{x^2+y^2}$, so $\mathbf{V} = \omega\sqrt{x^2+y^2}\mathbf{T}$, where $\mathbf{T}$ is the unit tangent to the circle.The vector $\omega\mathbf{K}\times\mathbf{X} = \omega\mathbf{K}\times(x\mathbf{I}+y\mathbf{J}+z\mathbf{K}) = \omega(x\mathbf{J}-y\mathbf{I}) = \omega\sqrt{x^2+y^2}\mathbf{T}$ also.

b) Calculate div $\mathbf{V}$, curl $\mathbf{V}$.

Answer. div$\mathbf{V}=0$ and curl $\mathbf{V}$ = curl $\omega(x\mathbf{J}-y\mathbf{I}) = 2\omega\mathbf{K}$.

c) Calculate $\int_C \mathbf{V}\cdot d\mathbf{X}$, where C is the helicoid given parametrically by $\mathbf{X}(t) = 2\cos t\mathbf{I}+2\sin t\mathbf{J}+t\mathbf{K}$.

Answer.

$$\int_C \mathbf{V}\cdot d\mathbf{X} = \int_C \omega(-y\mathbf{I}+x\mathbf{J})\cdot(dx\mathbf{I}+dy\mathbf{J}+dz\mathbf{K})$$
$$= \omega\int_C -ydx+xdy = \omega\int(\sin^2 t+\cos^2 t)dt = \omega\int dt\ ,$$

which is ω times the difference between the ending and initial times.

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2. Let $\mathbf{F}(\mathbf{X})$ be the vector field

$$\cos x\cos y(cosz+2)\mathbf{I} - (\sin x\sin y(2+\cos z)+\cos y\cos z)\mathbf{J} + \sin z(\sin y - \sin x\cos y)\mathbf{K}\ .$$

If it exists, find $w = f(x,y,z)$ such that $\nabla w = \mathbf{F}$.

Answer. First we find the general function f such that $\partial f/\partial x = P$:

$$f(x,y,z) = \sin x\cos y(cosz+2)+\phi(y,z)\ ,$$

where ϕ is an arbitrary function of y and z. Now we check for $\partial f/\partial y = Q$:

$$\frac{\partial f}{\partial y} = -\sin x\sin y(\cos z+2)+\frac{\partial\phi}{\partial y} = -(\sin x\sin y(2+\cos z)+\cos y\cos z)\ ,$$

giving us

$$\frac{\partial\phi}{\partial y} = -\cos y\cos z\ .$$

Integrating, $\phi = -\sin y\cos z+\psi(z)$ for some function ψ, and we now know this much about f:

$$f(x,y,z) = \sin x\cos y(\cos z+2)-\sin y\cos z+\psi(z)\ .$$

Differentiating with respect to z and setting equal to R gives us

$$\frac{\partial f}{\partial z} = -\sin x\cos y\sin z+\sin y\sin z+\psi'(z) = \sin z(\sin y-\sin x\cos y)\ .$$

Thus $\psi' = 0$, so $\psi(z) = C$, and the answer is

$$f(x,y,z) = \sin x \cos y(\cos z + 2) - \sin y \cos z + C \ .$$

Note that we did not have to verify that curl $\mathbf{F} = \mathbf{0}$, for if that weren't true, the argument would have broken down for obvious reasons.

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3. A **radial** field is a field of the form $\mathbf{F}(\mathbf{X}) = g(\rho)\mathbf{X}$, where $\rho = |\mathbf{X}|$. Show that a radial field has a potential function; that is, there is a function $w = G(\mathbf{X})$ such that $\mathbf{F} = \nabla G$.

Answer. First, we calculate that $\nabla\rho = \mathbf{X}/\rho$, (remember $\rho = |\mathbf{X}| = \sqrt{x^2+y^2+z^2}$), so that

$$\mathbf{F}(\mathbf{X}) = g(\rho)\rho\nabla\rho \ .$$

Now, by the Chain rule, if $G(\rho) = \int g(\rho)\rho d\rho$, then $\mathbf{F}(\mathbf{X}) = \nabla G(|\mathbf{X}|)$.

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4. Here are two vector fields: (A) $\mathbf{F}(x,y,z) = (2xy^2z + yz^2)\mathbf{I} + (2x^2yz + xz^2)\mathbf{J} + (x^2y^2 + 2xyz)\mathbf{K}$

(B) $\mathbf{G}(x,y,z) = (2xy^2z + yz^2)\mathbf{I} + (x^2z + xz)\mathbf{J} + (x^2y^2 + xy)\mathbf{K}$

a) Which of the vector fields $\mathbf{F}$ and $\mathbf{G}$ has a chance of being a gradient? Why?

Answer. Using the symbolism $P\mathbf{I} + Q\mathbf{J} + R\mathbf{K}$ for the vector field, we find for $\mathbf{F}$:

$$P_y = 4xyz + z^2 = Q_x \ , \quad P_z = 2xy^2 + 2yz = R_x \ , \quad Q_z = 2x^2y + 2xz = R_y$$

so curl $\mathbf{F} = 0$, so it could be a gradient. However, for $\mathbf{G}$,

$$P_y = 4xyz + z^2 \ , \quad Q_x = 2xz + z$$

so $\mathbf{G}$ cannot be a gradient.

b) Pick one of your answers to a) and find the function f whose gradient is that field.

Answer. If $\mathbf{F} = \nabla f$, then

$$\frac{\partial f}{\partial x} = 2xy^2z + yz^2$$

so $f = x^2y^2z + xyz^2 + \phi(x,y)$. Setting

$$\frac{\partial f}{\partial y} = 2x^2yz + xz^2 + \frac{\partial \phi}{\partial y}$$

equal to the component of $\mathbf{J}$ in $\mathbf{F}$, we see that $\partial\phi/\partial y = 0$, so ϕ is independent of y. Doing the same with $\partial\phi/\partial z$ we see that ϕ is also independent of z. Thus $f(x,y,z) = x^2y^2z + xyz^2$.

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5. Describe the equipotentials of these vector fields $\mathbf{F}$:

$$a)\ y\mathbf{I} + x\mathbf{J} \ , \quad b)\ y\mathbf{I} - x\mathbf{J} \ , \quad c)\ x\mathbf{I} + 2y\mathbf{J} + z\mathbf{K} \ , \quad d)\ y\mathbf{I} + x\mathbf{J} - 2z\mathbf{K} \ .$$

Answer.

a) Here $\mathbf{F} = \nabla(xy)$, so $\mathbf{F}$ is orthogonal to the curves $xy =$ const. so the equipotentials of $\mathbf{F}$ are these hyperbolas.

b) Here curl $\mathbf{F} \neq 0$; however, $\mathbf{F}$ is orthogonal to the surfaces $x/y =$ const., since

$$\nabla(\frac{x}{y}) = \frac{1}{y}\mathbf{I} - \frac{x}{y^2}\mathbf{J} = \frac{y\mathbf{I} - x\mathbf{J}}{y^2} = \frac{1}{y^2}\mathbf{F} \, ,$$

So $\nabla(x/y)$ and $\mathbf{F}$ are collinear. This example shows that a vector field can have equipotentials even if its curl is nonzero. Given a vector field $\mathbf{F}$, a function μ is called an **integrating factor** for $\mathbf{F}$ if curl $\mu\mathbf{F} = 0$. If a vector field has an integrating factor, it has equipotentials.

c) Here $\mathbf{F} = \nabla((x^2 + 2y^2 + z^2)/2)$, so the euipotentials of $\mathbf{F}$ are the ellipsoids $x^2 + 2y^2 + z^2 =$ const.

d) Here $\mathbf{F} = \nabla(xy - z^2)$, so the equipotentials of $\mathbf{F}$ are the hyperboloids $xy - z^2 =$ const.

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6. Let C be the curve $x = y = z$ going from the point (1,1,1) to the point $(4,4,4)$. Find

$$\int_C \mathbf{F} \cdot d\mathbf{X}, \qquad \int_C \mathbf{G} \cdot d\mathbf{X}$$

for the vector fields given in (A) and (B) of problem 4.

Answer. Since $\mathbf{F} = \nabla f = \nabla(x^2y^2z + xyz^2)$, the line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \int_C \nabla f \cdot d\mathbf{X} = \int_0^4 \frac{df}{dt} dt = f(4,4,4) - f(1,1,1) = 4^5 + 4^4 - (1+1) = 4^4(5) - 2 \, .$$

For the second integral,we'll have to calculate. We take the parametrization of C : $\mathbf{X}(t) = t\mathbf{I} + t\mathbf{J} + t\mathbf{K}$, $1 \leq t \leq 4$. Then $d\mathbf{X} = (\mathbf{I} + \mathbf{J} + \mathbf{K})dt$ and

$$\int_C \mathbf{G} \cdot d\mathbf{X} = \int_1^4 (2t^4 + t^3 + t^3 + t^2 + t^4 + t^2)dt = \int_1^4 (3t^4 + 2t^3 + 2t^2)dt = 4^3(\frac{76}{15}) - \frac{53}{30}$$

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7. Consider the force field in three dimensions $\mathbf{F}(x,y,z) = y\mathbf{I} + z\mathbf{K}$. Let C be the curve given parametrically by $\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + t\mathbf{K}$. What is the work required to move a particle along C from $(1,0,0)$ to $(1,0,10\pi)$?

Answer. Compute: $d\mathbf{X} = (-\sin t\mathbf{I} + \cos t\mathbf{J} + \mathbf{K})dt$, so

$$\int_C \mathbf{F} \cdot \mathbf{X} = \int_0^{10\pi} (-\sin^2 t + t)dt$$

$$= \int_0^{10\pi} \big[-(\frac{1 - \cos 2t}{2}) + t\big]dt = -10\pi/2 + 100\pi^2/2 = -5\pi(1 + 10\pi) \, .$$

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8. Consider the vector field defined in the firt quadrant by

$$\mathbf{F}(x,y) = (\frac{y}{x} + \ln y)\mathbf{I} + (\frac{x}{y} + \ln x)\mathbf{J} \, .$$

Find $\int_C \mathbf{F} \cdot d\mathbf{X}$ where C is the straight line from (1,1) to (e, e^2).

Answer. $\mathbf{F} = \nabla f$, where $f(x,y) = y\ln x + x\ln y$. Thus $\int_C \mathbf{F} \cdot d\mathbf{X} = f(e,e^2) - f(1.1) = e^2 \ln e + e\ln(e^2) = e^2 + 2e$. If you don't observe that $\mathbf{F}$ is a gradient, you have a long calculation to do.