Calculus III
Practice Problems 10

1. Suppose that a fluid is rotating about the z-axis with constant angular speed ω. Let $\mathbf{V}$ be the velocity field of the motion.

a) Show that $\mathbf{V} = \omega\mathbf{K} \times \mathbf{X}$.

b) Calculate div $\mathbf{V}$, curl $\mathbf{V}$.

c) Calculate $\int_C \mathbf{V} \cdot d\mathbf{X}$, where C is the helicoid given parametrically by $\mathbf{X}(t) = 2\cos t\mathbf{I} + 2\sin t\mathbf{J} + t\mathbf{K}$.

2. Let $\mathbf{F}(\mathbf{X})$ be the vector field

$$\cos x \cos y(cosz + 2)\mathbf{I} - (\sin x \sin y(2 + \cos z) + \cos y \cos z)\mathbf{J} + \sin z(\sin y - \sin x \cos y)\mathbf{K} \ .$$

If it exists, find $w = f(x,y,z)$ such that $\nabla w = \mathbf{F}$.

3. A **radial** field is a field of the form $\mathbf{F}(\mathbf{X}) = g(\rho)\mathbf{X}$, where $\rho = |\mathbf{X}|$. Show that a radial field has a potential function; that is, there is a function $w = G(\mathbf{X})$ such that $\mathbf{F} = \nabla G$.

4. Here are two vector fields:

$$(A) \qquad \mathbf{F}(x,y,z) = (2xy^2z + yz^2)\mathbf{I} + (2x^2yz + xz^2)\mathbf{J} + (x^2y^2 + 2xyz)\mathbf{K}$$

$$(B) \qquad \mathbf{G}(x,y,z) = (2xy^2z + yz^2)\mathbf{I} + (x^2z + xz)\mathbf{J} + (x^2y^2 + xy)\mathbf{K}$$

a) Which of the vector fields $\mathbf{F}$ and $\mathbf{G}$ has a chance of being a gradient? Why?

b) Pick one of your answers to a) and find the function f whose gradient is that field.

5. Describe the equipotentials of these vector fields $\mathbf{F}$:

$$a)\ y\mathbf{I} + x\mathbf{J}\ , \quad b)\ y\mathbf{I} - x\mathbf{J}\ , \quad c)\ x\mathbf{I} + 2y\mathbf{J} + z\mathbf{K}\ , \quad d)\ y\mathbf{I} + x\mathbf{J} - 2z\mathbf{K}\ .$$

6. Let C be the curve $x = y = z$ going from the point (1,1,1) to the point $(4,4,4)$. Find

$$\int_C \mathbf{F} \cdot d\mathbf{X}, \qquad \int_C \mathbf{G} \cdot d\mathbf{X}$$

for the vector fields given in (A) and (B) of problem 4.

7. Consider the force field in three dimensions $\mathbf{F}(x,y,z) = y\mathbf{I} + z\mathbf{K}$. Let C be the curve given parametrically by $\mathbf{X}(t) = \cos t\mathbf{I} + \sin t\mathbf{J} + t\mathbf{K}$. What is the work required to move a particle along C from $(1,0,0)$ to $(1,0,10\pi)$?

8. Consider the vector field defined in the firt quadrant by

$$\mathbf{F}(x,y) = (\frac{y}{x} + \ln y)\mathbf{I} + (\frac{x}{y} + \ln x)\mathbf{J}\ .$$

Find $\int_C \mathbf{F} \cdot d\mathbf{X}$ where C is the straight line from (1,1) to (e, e^2).