Calculus III
Practice Problems 1: Answers

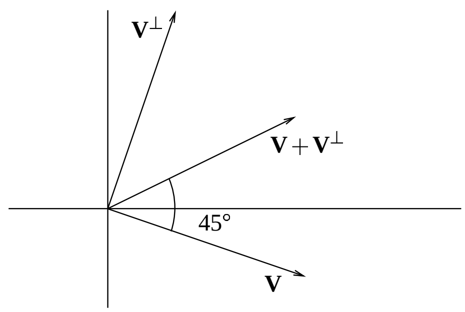
1. Find the components of a vector $\mathbf{V}$ making an angle of $45°$ with $\mathbf{W} = 2\mathbf{I} - \mathbf{J}$.

Answer. Let α be the angle $\mathbf{W}$ makes with the horizontal. We have $\tan\alpha = -1/2$, so $\alpha = -26.6°$. Thus the vector $\mathbf{V}$ makes an angle of $18.4°$ with the horizontal, and we can take

$$\mathbf{V} = (\cos 18.4°)\mathbf{I} + (\sin 18.4°)\mathbf{J} = .949\mathbf{I} + .316\mathbf{J}$$

Here's another solution. Since $\mathbf{W}^{\perp}$ has the same length as $\mathbf{W}$ and is orthogonal to $\mathbf{W}$, we can take $\mathbf{V}$ as the diagonal of the rectangle spanned by $\mathbf{W}$ and $\mathbf{W}^{\perp}$ (see the figure).

Thus, we take $\mathbf{V} = \mathbf{W} + \mathbf{W}^{\perp} = 3\mathbf{I} + \mathbf{J}$. Note that the first solution is the unit vector in the direction of $\mathbf{W} + \mathbf{W}^{\perp}$.



---

2. Show that the diagonals of a rhombus intersect at right angles. (A *rhombus* is a parallelogram with all sides of the same length).

Answer. Let $\mathbf{V}$ and $\mathbf{W}$ be the sides of the rhombus. The diagonals are $\mathbf{V} + \mathbf{W},\ \mathbf{V} - \mathbf{W}$. To show the diagonals are orthogonal, we calculate

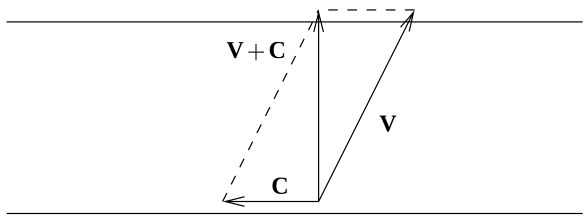
$$(\mathbf{V} + \mathbf{W}) \cdot (\mathbf{V} - \mathbf{W}) = \mathbf{V} \cdot \mathbf{V} - \mathbf{V} \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{W} = |\mathbf{V}|^2 - |\mathbf{W}|^2 = 0$$

since $\mathbf{V}$ and $\mathbf{W}$ have the same length.

---

3. A river ferry runs at a speed of 6 knots, across a river with a current of 2 knots. Assume that the river has shores which are parallel straight lines. In what direction should the barge head in order to cross the river perpendicular to the shores?

Answer. The ferry should head upstream at some angle α to the vertical, so that the resultant of the ferry velocity $\mathbf{V}$ and the stream velocity $\mathbf{C}$ is directly vertical. (See the figure). Since $\mathbf{C}$ is horizontal of magnitude 2, $\mathbf{C} = 2\mathbf{I}$, and since $|\mathbf{V}| = 6$, $\mathbf{V} = -6\sin\alpha\mathbf{I} + 6\cos\alpha\mathbf{J}$. For $\mathbf{V} + \mathbf{C}$ to be a multiple of $\mathbf{J}$, we must have $-6\sin\alpha = 2$, so $\sin\alpha = 1/3$, and $\alpha = 19.47°$.



4. Find the area of the parallelogram determined by the vectors $\mathbf{V} = 6\mathbf{I} - 7\mathbf{J}$, $\mathbf{W} = 3\mathbf{I} + 4\mathbf{J}$. What are the coordinates of the vertex of this parallelogram farthest from the origin?

Answer. The area is $|\det(\mathbf{V}, \mathbf{W})| = |6(4) - (-7)(3)| = 45$. The vector $\mathbf{V} + \mathbf{W} = \mathbf{9I} - \mathbf{3J}$ goes from the origin to the furthest vertex, so that vertex has coordinates (9,-3).

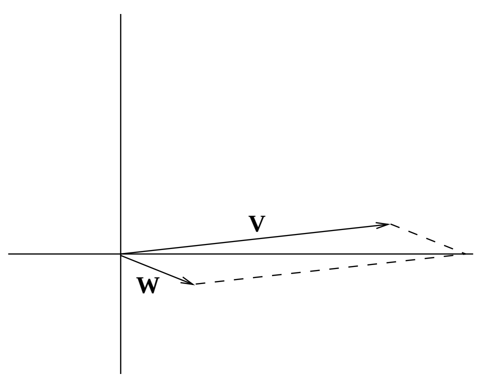
5. A plane flies at a ground speed of 480 mph. The jet stream comes from 30° north of west at 60 mph. In what direction should the pilot direct the plane so as to be heading directly east? How many miles will the plane cover in 2 hours?
Answer. Let $\mathbf{W}$ be the velocity vector of the wind. Then, by the figure, we have

$$\mathbf{W} = 60\left(\frac{\sqrt{3}}{2}\mathbf{I} - \frac{1}{2}\mathbf{J}\right) .$$

Let $\mathbf{V} = a\mathbf{I} + b\mathbf{J}$ be the velocity vector of the plane (relative to ground). The information we have is that $|\mathbf{V}| = 480$, and the sum $\mathbf{V} + \mathbf{W}$ point in the direction of $\mathbf{I}$, so its $\mathbf{J}$-component is zero. This gives the equation $b - 30 = 0$, so $b = 30$. Then $a^2 + 30^2 = 480^2$, so $a = 479.1$. Then

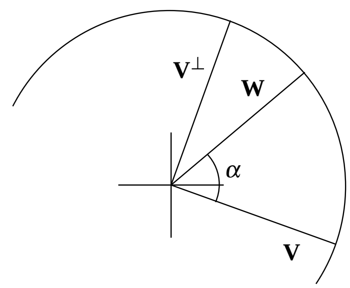
$$\mathbf{V} = 479.1\mathbf{I} + 30\mathbf{J} , \quad \text{and} \quad \mathbf{V} + \mathbf{W} = (479.1 + 30\sqrt{3})\mathbf{I} = 531\mathbf{J} .$$

In two hours the plane travels 1062 miles.



6. Let $\mathbf{V} = 2\mathbf{I} - 3\mathbf{J}$. Find the vector $\mathbf{W}$ to the left of $\mathbf{V}$, of the same length as $\mathbf{V}$, and making an angle of 60° with $\mathbf{V}$.

Answer. Recall that, for any vector $\mathbf{V}$, $\mathbf{V}^\perp$ is perpendicular to, and left of $\mathbf{V}$, and of the same length. Thus, any vector of the form $\mathbf{W} = \cos\alpha\mathbf{V} + \sin\alpha\mathbf{V}^\perp$ has the same length as $\mathbf{V}$ and $\mathbf{V}^\perp$ and makes an angle α with $\mathbf{V}$.



We can see this from the figure above, but it also can be computed:

$$|\mathbf{W}|^2 = \cos^2\alpha|\mathbf{V}|^2 + 2\cos\alpha\sin\alpha\mathbf{V}\cdot\mathbf{V}^\perp + \sin^2\alpha|\mathbf{V}^\perp|^2 = |\mathbf{V}|^2 \ ,$$

since $\mathbf{V}\cdot\mathbf{V}^\perp = \mathbf{0}$, and

$$\mathbf{W}\cdot\mathbf{V} = |\mathbf{V}|^2\cos\alpha = |\mathbf{W}||\mathbf{V}|\cos\alpha \ ,$$

so α is the angle between $\mathbf{W}$ and $\mathbf{V}$. In our case, $\mathbf{V}^\perp = 3\mathbf{I} + 2\mathbf{J}$ and $\alpha = \pi/3$, so the answer is

$$\mathbf{W} = \cos\left(\frac{\pi}{3}\right)\mathbf{V} + \sin\left(\frac{\pi}{3}\right)\mathbf{V}^\perp = \left(\frac{2+3\sqrt{3}}{2}\right)\mathbf{I} + \left(\frac{-3+2\sqrt{3}}{2}\right)\mathbf{J} \ .$$

---

7. Find a base $\mathbf{L}$, $\mathbf{M}$ (in a base, the vectors are of length 1 and orthogonal) so that $\mathbf{L}$ is on the line $y = 5x$ and $\mathbf{M}$ is left of $\mathbf{L}$. Any vector $\mathbf{X} = x\mathbf{I} + y\mathbf{J}$ can be written in the form $\mathbf{X} = u\mathbf{L} + v\mathbf{M}$. Find u, v as functions of x, y.

Answer. A unit vector on the line $y = 5x$ is $\mathbf{L} = (\mathbf{I} + 5\mathbf{J})/\sqrt{26}$. Thus

$$\mathbf{L} = \frac{\mathbf{I} + 5\mathbf{J}}{\sqrt{26}} \ , \quad \mathbf{M} = \mathbf{L}^\perp = \frac{-5\mathbf{I} + \mathbf{J}}{\sqrt{26}} \ .$$

For $\mathbf{X} = x\mathbf{I} + y\mathbf{J} = u\mathbf{L} + v\mathbf{M}$, we have

$$u = \mathbf{X}\cdot\mathbf{L} = \frac{x + 5y}{\sqrt{26}} \ , \quad v = \mathbf{X}\cdot\mathbf{M} = \frac{-5x + y}{\sqrt{26}} \ .$$

---

8. What is the distance between the two parallel lines $L_1 : x + 2y = 7$, $L_2 : x + 2y = 11$?

Answer. Let $\mathbf{V}$ be the vector on the y-axis joining L_1 to L_2. Then $\mathbf{V} = 2\mathbf{J}$, and the distance we seek is the length of the projection of $\mathbf{V}$ in the direction orthogonal to the lines. The vector $\mathbf{N} = \mathbf{I} + 2\mathbf{J}$ is orthogonal to the lines, thus

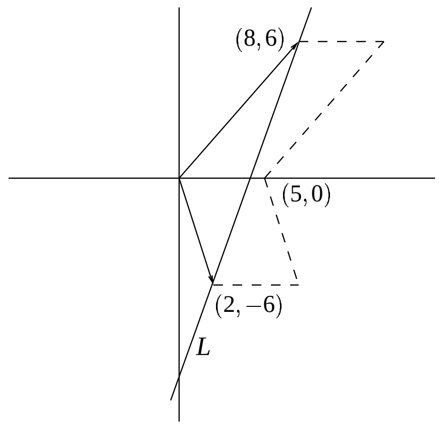
$$d = \frac{\mathbf{V}\cdot\mathbf{N}}{|\mathbf{N}|} = \frac{4}{\sqrt{5}} \ .$$

---

9. Find the distance from the point P(3,-2) to the line $L : 2x - y = 10$.

Answer. Since Q(5,0) is on the line L, the vector $\overrightarrow{PQ} = 2\mathbf{I} + \mathbf{J}$ joins P to a point in Q. Thus, the distance sought is the length of the projection of $\overrightarrow{PQ}$ in the direction normal to L. The vector $\mathbf{N} = 2\mathbf{I} - 2\mathbf{J}$ points in this direction. Thus

$$d = \frac{\overrightarrow{PQ} \cdot \mathbf{N}}{|\mathbf{N}|} = \frac{(2\mathbf{I} + \mathbf{J}) \cdot (2\mathbf{I} - 2\mathbf{J})}{2\sqrt{2}} = \frac{1}{\sqrt{2}} .$$



10. Find a point (x, y) on the line $2x - y = 10$ such that the triangle with vertices (0,0), (5,0), (x, y) has area equal to 15. How many such points are there?

Answer. Let P be the point; since it is on the given line, $y = 2x - 10$. The hypothesis tells us that the vectors $x\mathbf{I} + (2x - 10)\mathbf{J}$ and $5\mathbf{I}$ must span a parallelogram of area 30. Thus

$$|\det(x\mathbf{I} + (2x - 10)\mathbf{J}, 5\mathbf{I}| = 30 \quad \text{which gives us} \quad |(-5)(2x - 10)| = 30 ,$$

or, the two equations $2x - 10 = 6,\ 2x - 10 = -6$. Thus the possible solutions for P are (8,6), (2,-6).