MATH 2210-90 Fall 2011

Third Midterm Exam

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LAST NAME ____________________

FIRST NAME Grader's Copy

ID NO. ____________________

INSTRUCTION: SHOW ALL OF YOUR WORK. MAKE SURE YOUR ANSWERS ARE CLEAR AND LEGIBLE. USE **SPECIFIED** METHOD TO SOLVE THE QUESTION. IT IS NOT NECESSARY TO SIMPLIFY YOUR FINAL ANSWERS.

PROBLEM 1	20	______
PROBLEM 2	20	______
PROBLEM 3	20	______
PROBLEM 4	20	______
PROBLEM 5	20	______
TOTAL	100	______

PROBLEM 1

(20 pt) Evaluate the integral

$$\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx.$$

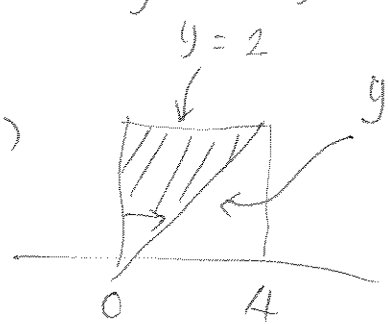
$\frac{x}{2} =$ $\quad y = 0$

(1) e^{y} has no "closed" integal form, i.e. it is impossible to find a "proper" antiderivative

(5 pt)

(2) Chage dy dx to dx dy

(5 pt)



$y = \frac{x}{2}$ equivalent

$y = 0$ to 2

$x = 0$ to $2y$

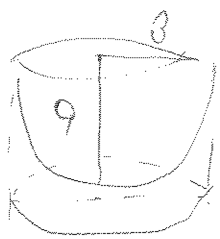
(10 pt)

$$\int_{y=0}^{2} \int_{x=0}^{2y} e^{y^2}\, dx\, dy$$

$$= \int_{y=0}^{2} 2y\, e^{y^2}\, dy = e^{y^2} \Big|_0^2$$

PROBLEM 3

(20 pt) Find the area of the surface $z = x^2 + y^2$ below the plane $z = 9$.



surface differential

$$dA = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \qquad (5pt)$$

$$= \sqrt{(2x)^2 + (2y)^2 + 1}$$

~~polar~~

$$= \sqrt{4r^2 + 1} \qquad (5pt)$$

Symmetry

(5pt)

$$A = \int_{\theta=0}^{2\pi} \int_{r=0}^{3} \sqrt{4r^2+1} \; r \, dr \, d\theta$$

$$= 2\pi \left[\frac{1}{8} \frac{2}{3} (4r^2+1)^{\frac{3}{2}} \right] \Big|_{0=r}^{3}$$

(5pt)

PROBLEM 2

(20 pt) Using polar coordinates, evaluate the integral

$$\int\int_R e^{-x^2}e^{-y^2}dA,$$

where R is the xy- plane, $\mathbb{R}^2$.
(10 pt) **Bonus:** Find

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx.$$

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA \quad (5pt)$$

$$= \int_{\mathbb{R}^2} e^{-r^2}\, r\, dr\, d\theta \quad (5pt)$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{\infty} r e^{-r^2}\, dr \right) d\theta \quad (5pt)$$

$$= 2\pi \left(-\frac{1}{2}\right) e^{-r^2} \Big|_0^{\infty} \quad [\text{improper L'Hopital}]$$

$$= \pi \quad (5pt)$$

Bonus: $\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \right]^2$ (5pt)

so $\int \cdots = 1$

$$= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta \quad (5pt)$$

$$= 2\pi \left(\frac{1}{\sqrt{2\pi}}\right)^2 e^{-\frac{r^2}{2}} \Big|_0^{\infty} = 1$$

PROBLEM 5

(20 pt) Derive the change of variable formula

$$dx\ dy\ dz = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

for the transformation from Cartesian coordinates to spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta \qquad (5pt)$$

$$z = \rho \cos\phi$$

$$J = \begin{vmatrix} \cos\theta \sin\phi & \rho \cos\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \sin\phi \sin\theta & \rho \cos\phi \sin\theta & \rho \sin\phi \cos\theta \\ \cos\phi & -\rho \sin\phi & 0 \end{vmatrix} \qquad (5pt)$$

$$\det = \cos\phi \begin{vmatrix} \rho \cos\phi \cos\theta & -\rho \sin\phi \sin\theta \\ \rho \cos\phi \sin\theta & \rho \sin\phi \cos\theta \end{vmatrix}$$

(10pt)

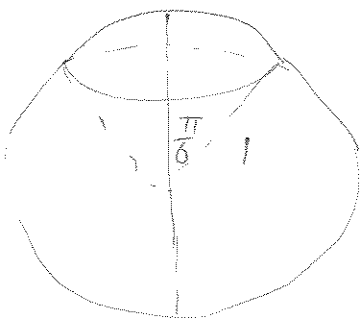
$$+ \rho \sin\phi \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\phi \sin\theta \\ \sin\theta \sin\phi & \rho \sin\phi \cos\theta \end{vmatrix}$$

$$= \cos\phi \cdot \rho^2 \cos\phi \sin\phi + \rho^2 \sin^3\phi$$

$$= \rho^2 \sin\phi$$

PROBLEM 4

(20 pt) Find the volume and the center of mass of the homogeneous solid S that is bounded above by the sphere $\rho = 1$ and below by the cone $\phi = \pi/6$.



$$\theta \Big|_0^{2\pi} \ \frac{1}{3}\rho^3 \Big|_0^1 \left(-\cos\phi\right)\Big|_0$$

(5pt)

$$V = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=0}^{\frac{\pi}{6}} \sin\phi \ \rho^2 \, d\phi \, d\rho \, d\theta$$

$$= 2\pi \cdot \frac{1}{3} \left[1 - \frac{\sqrt{3}}{2}\right]$$ (5pt)

Center : due to symmotry

it is on tho z-axis

important
$z = \rho\cos\phi$

weighted $V_z = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \int_{\phi=0}^{\frac{\pi}{6}} \cos\phi \sin\phi \ \rho^3 \, d\phi \, d\rho \, d\theta$

(~~5~~ 10 pt) $= 2\pi \cdot \frac{1}{4} \cdot \frac{1}{2} \sin^2\phi \Big|_0^{\frac{\pi}{6}}$

$= 2\pi \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4}$

Subsitution : $u = \sin\phi$

Center : $(0, 0, [3/16(2-\sqrt{3})])$