PROBLEM 1

(30 pt) Let T be the triangle with vertices at $P = (-8, -9)$, $Q = (-7, 10)$, $R = (-7, 1)$, find the area of T.

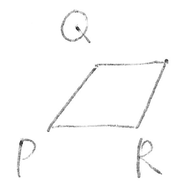
$\vec{PR} = (1, 10)$

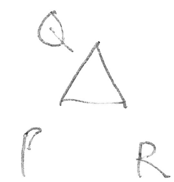
$\vec{PQ} = (1, 19)$ (10pt)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 10 & 0 \\ 1 & 19 & 0 \end{vmatrix} = -29\,\hat{k} \quad \text{(10pt)}$$

$$= \vec{PR} \times \vec{PQ}$$

(10pt)

$|-29|$ = area of 

$|-29|/2 = 29/2$ = Area of 

(10pt)

PROBLEM 2

(30 pt) If $w = x^2 + y^2 + z^2 + xy$, where $x = st$, $y = s - t$, $z = s + 2t$, find $\partial w/\partial s$.

(10pt)

$$\partial w/\partial s = \left(2x \frac{\partial x}{\partial s} + 2y \frac{\partial y}{\partial s} + 2z \frac{\partial z}{\partial s}\right) + \left(\frac{\partial x}{\partial s} y + x \frac{\partial y}{\partial s}\right)$$

(10pt)

$$\frac{\partial x}{\partial s} = \frac{\partial y}{\partial s} = \frac{\partial z}{\partial s} = 1$$

(10 pt)

PROBLEM 3

(30 pt) Evaluate the integral

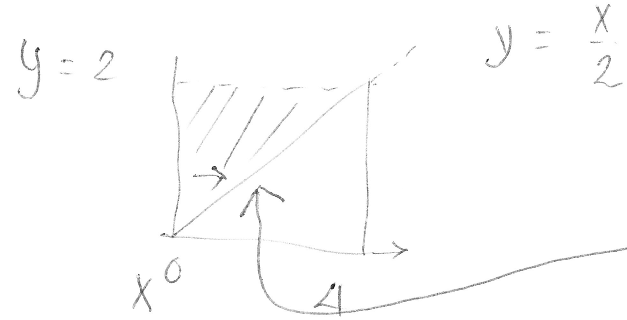
$$\int_0^4 \int_{x/2}^2 e^{y^2}\, dy dx.$$

$\int e^{y^2} dy$ won't yield an

(5pt) algebraic closed form

(i.e. you cannot find the "nice"

fcn for the integral)



(5pt)

$$\int_{y=0}^{2} \int_{x=0}^{2y} e^{y^2}\, dx\, dy$$

(10pt)

$$= \int_{y=0}^{2} e^{y^2} \cdot 2y\, dy$$

$$= e^{y^2} \Big|_0^2$$ (10pt)

PROBLEM 5

(30 pt)

$$\mathbf{F}(x,y) = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

Let C be the circle $\mathbf{X}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \le t \le 2\pi$. Compute $\int_C \mathbf{F} \cdot d\mathbf{X}$.

$\vec{X}(t) = x(t)\,i + y(t)\,j$

i.e. $x(t) = \cos(t)$

$y(t) = \sin(t)$

Note.

$x'(t) = -y(t)$

$y'(t) = x(t)$

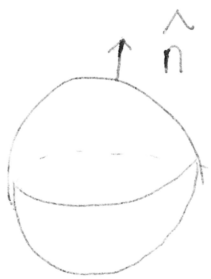
$$\int \left(\frac{-y(t)}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \cdot (dx, dy)$$

$$\int \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \cdot (-y, x)\, dt$$

$$= \int_{t=0}^{2\pi} \frac{y^2+x^2}{x^2+y^2}\, dt = 2\pi$$

PROBLEM 6

(30 pt) Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ and S be the sphere $x^2 + y^2 + z^2 = 9$. Denote by $\mathbf{n}$ the outward unit normal vector to S. Compute $\int_S \mathbf{F}\cdot\mathbf{n}ds$.



(10pt)

Gauss $= \int \nabla\cdot\vec{F} \; dV$

$= \int (0 + 0 + 2)\; dV$ (5pt)

$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_{\rho=0}^{3} 2\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$ (5pt)

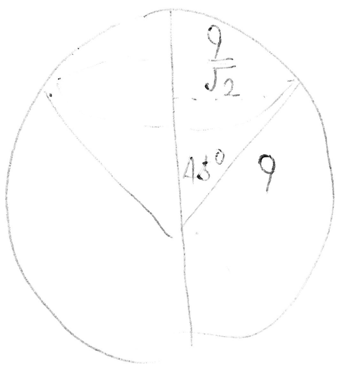
$= \frac{8}{3}\pi \cdot 3^3$

(10pt)

PROBLEM 7

(30 pt) Find the surface area of the part of the sphere $x^2+y^2+z^2=81$ that lies above the cone $z=\sqrt{x^2+y^2}$.

method (1)



$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{4}} 9^2 \sin\phi \, d\phi \, d\theta$$

(10pt)

$$= 9^2 \cdot 2\pi \cdot (-\cos\phi)$$

(10pt)

$$= 9^2 \cdot 2\pi \left(1-\frac{\sqrt{2}}{2}\right)$$

(10pt)

method (2) Surface integral (10 pt)

$f(x,y,z) = x^2+y^2+z^2-81=0 \qquad z=\sqrt{81-x^2-y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{81-x^2-y^2}}$$

(10pt)

$$S = \int \sqrt{1+z_x^2+z_y^2}\, dA \qquad z_y = \frac{-y}{\sqrt{81-x^2-y^2}}$$

$$= 2\pi \int_{r=0}^{\frac{9}{\sqrt{2}}} \frac{9}{\sqrt{9-r^2}}\, r\, dr = 2\pi \cdot 9 \cdot \sqrt{9-r^2}\Big|_{9/\sqrt{2}}^{0}$$

(10pt)

PROBLEM 8

(10 pt) The unit disc in the plane has boundary S: the circle, $x^2+y^2 = 1$. Find out

$$\frac{1}{2}\oint_S (-ydx + xdy).$$

Method 1. Green's

$M = -y \qquad N = x \qquad N_x = 1$

$M_y = 1$

(5pt)

$\text{Int} = \frac{1}{2} \int_D 2 \, dA$

↑ unit disk

(5pt)

$= \frac{1}{2} \cdot 2 \cdot \pi = \pi$

Method 2

$x = \cos\theta \qquad y = \sin\theta$

$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta$

$\frac{1}{2}\int_{\theta=0}^{2\pi} (+\cos\theta)(\cos\theta)$

$(-\sin\theta)(-\sin\theta)\, d\theta$

$= \frac{1}{2} \cdot 2\pi = \pi$