Mathematics 2210 Calculus III
Practice Final Examination

1. Find the symmetric equations of the line through the point (3,2,1) and perpendicular to the plane $7x - 3y + z = 14$.

Solution. The vector $\mathbf{V} = 7\mathbf{I} - 3\mathbf{J} + \mathbf{K}$ is orthogonal to the given plane, so points in the direction of the line. If we let $\mathbf{X}_0 = 3\mathbf{I} + 2\mathbf{J} + \mathbf{K}$, then the condition for $\mathbf{X}$ to be the vector to a point on the line is $\mathbf{X} - \mathbf{X}_0$ is collinear with $\mathbf{V}$, which gives us the symmetric equations

$$\frac{x-3}{7} = \frac{y-2}{-3} = \frac{z-1}{1} \ .$$

2. Find the equation of the plane through the points (0,-1,1), (1,0,1) and (1,2,2).

Solution. The vectors from the first point (call it $\mathbf{X}_0$) to the second and third points are $\mathbf{U} = \mathbf{I} + \mathbf{J}$, $\mathbf{V} = \mathbf{I} + 3\mathbf{J} + \mathbf{K}$. Since $\mathbf{U}, \mathbf{V}$ lie on the plane $\mathbf{U} \times \mathbf{V}$ is normal to the plane. We calculate $\mathbf{N} = \mathbf{U} \times \mathbf{V} = \mathbf{I} - \mathbf{J} + 2\mathbf{K}$. Thus the equation of the plane is

$$\mathbf{X} \cdot \mathbf{N} = \mathbf{X}_0 \cdot \mathbf{N} \ , \quad \text{or} \quad x - y + 2z = 3 \ .$$

3. A particle moves through the plane as a function of time: $\mathbf{X}(t) = t^2\mathbf{I} + 2t^3\mathbf{J}$. Find the unit tangent and normal vectors, and the tangential and normal components of the acceleration.

Solution. $\mathbf{V} = 2t\mathbf{I} + 6t^2\mathbf{J}$, $\mathbf{A} = 2\mathbf{I} + 12t\mathbf{J}$. Thus $ds/dt = 2t\sqrt{1+9t^2}$ and

$$\mathbf{T} = \frac{\mathbf{I} + 3t\mathbf{J}}{\sqrt{1+9t^2}} \ , \quad \mathbf{N} = \frac{-3t\mathbf{I} + \mathbf{J}}{\sqrt{1+9t^2}} \ .$$

Then

$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{2 + 36t^2}{\sqrt{1+9t^2}} \ , \qquad a_N = \mathbf{A} \cdot \mathbf{N} = \frac{6t}{\sqrt{1+9t^2}} \ .$$

4. A particle moves through space as a function of time:

$$\mathbf{X}(t) = \cos t\mathbf{I} + t \sin t\mathbf{J} + t\mathbf{K} \ .$$

For this motion, find $\mathbf{T}$, $\mathbf{N}$, the the tangential and normal components of the acceleration, and the curvature at time $t = 3\pi/2$.

Solution. $\mathbf{V} = -\sin t\mathbf{I} + (\sin t + t\cos t)\mathbf{J} + \mathbf{K}$, $\mathbf{A} = -\cos t\mathbf{I} + (2\cos t - t\sin t)\mathbf{J}$. Evaluate at $t = 3\pi/2$:

$$\mathbf{V} = \mathbf{I} - \mathbf{J} + \mathbf{K} \ , \qquad \mathbf{A} = \frac{3\pi}{2}\mathbf{J} \ .$$

Then

$$a_T = \mathbf{A}\cdot\mathbf{T} = \mathbf{A}\cdot\frac{\mathbf{V}}{|\mathbf{V}|} = -\frac{\pi}{2}\sqrt{3}\ , \quad a_N\mathbf{N} = \mathbf{A} - a_T\mathbf{T} = \frac{\pi}{2}(\mathbf{I}+2\mathbf{J}+\mathbf{K})$$

so

$$a_N = \frac{\pi}{2}\sqrt{6}\ , \qquad \mathbf{N} = \frac{\mathbf{I}+2\mathbf{J}+\mathbf{K}}{\sqrt{6}}\ .$$

5. The particle of problem 3 moves in opposition to the force field $\mathbf{F}(x,y,z) = x\mathbf{I} - y\mathbf{J} - \mathbf{K}$. How much work is required to move the particle from (1,0,0) to $(1,0,2\pi)$?

Solution. The curve is parametrized by $\mathbf{X}(t) = \cos t\mathbf{I} + t\sin t\mathbf{J} + t\mathbf{K},\ 0 \le t \le 2\pi$, so along the curve

$$\mathbf{F} = \cos t\mathbf{I} - t\sin t\mathbf{J} - \mathbf{K}\ , \quad d\mathbf{X} = (-\sin t\mathbf{I} + (\sin t + t\cos t)\mathbf{J} + \mathbf{K})dt\ .$$

Thus the work is

$$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_0^{2\pi}(-\cos t\sin t - t\sin t(\sin t + t\cos t) - 1)dt = -2\pi\ .$$

This calculation can be avoided by noticing that $\mathbf{F}$ is a gradient field: $\mathbf{F} = \nabla f$, with $f(x,y,z) = (x^2 - y^2)/2 - z$. Thus

$$\int_C \mathbf{F}\cdot d\mathbf{X} = f(1,0,2\pi) - f(1,0,0) = -2\pi\ .$$

6. Find the critical points of

$$f(x,y) = 3xy + \frac{1}{x} - \ln y$$

in the first quadrant. Classify as local maximum or minimum or saddle point.

Solution. $\nabla f = (3y - 1/x^2)\mathbf{I} + (3x - 1/y)\mathbf{J}$. We solve $\nabla f = 0\ :\ 3y = x^{-2},\ 3x = y^{-1}$ give $x^2 = x$, so $x = 0$ or $x = 1$. Since $x = 0$ is not in the first quadrant, the only critical point is (1,1/3). At this point $f_{xx} = -2x^{-3} = -2,\ f_{yy} = y^{-2} = 9$, and $f_{xy} = 3$. Thus $D = (-2)(9) - 9 = -27$, and (1.1/3) is a saddle point.

7. The temperature distribution on the surface $x^2 + y^2 + z^2 = 1$ is given by $T(x,y,z) = xz + yz$. Find the hottest spot.

Solution. In the language of Lagrange multipliers, the objective function is $T(x,y,z) = xz + yz$, and the constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 1$. The gradients are $\nabla T = z\mathbf{I} + z\mathbf{J} + (x+y)\mathbf{K},\ \nabla g = 2(x\mathbf{I} + y\mathbf{J} + z\mathbf{K})$. The Lagrange equations are

$$z = \lambda x\ , \quad z = \lambda y\ , \quad x + y = \lambda z\ , \quad x^2 + y^2 + z^2 = 1\ .$$

Now, at $z = 0$, we have $T = 0$, so no such point is the hottest spot. The first equations therefore give us $x = y$. Replacing y by x we now have

$$z = \lambda x \ , \quad 2x = \lambda z \ , \quad 2x^2 + z^2 = 1 \ .$$

By the first two equations, $\lambda^2 = 2$, and the last equation then gives us $2x^2 + 2x^2 = 1$, so $x^2 = 1/4$. These then are the critical points:

$$x = y = \pm\frac{1}{2} \ , \quad z = \pm\frac{1}{\sqrt{2}} \ .$$

T takes its maximum at $(1/2, 1/2, 1/\sqrt{2})$, and its negative.

8. What is the equation of the tangent plane to the surface $z^2 - 3x^2 - 5y^2 = 1$ at the point (1,1,3)?

Solution. Take the differential of the defining equation of the surface: $2zdz - 6xdx + 10ydy = 0$. Substitute the coordinates of the point (1,1,3): $6dz - 6dx + 10dy = 0$. This is the equation of the tangent plane, with the differentials replaced by the increments:

$$6(z-3) - 6(x-1) + 10(y-1) \ , \quad \text{or} \quad -6x + 10y + 6z = 22 \ .$$

9. Consider the surface Σ

$$f(x,y) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1 \ .$$

a) At what points on Σ is the tangent plane parallel to the plane $2x + y - z = 1$?

Solution. The normal to the plane is $\mathbf{N} = 2\mathbf{I} + \mathbf{J} - \mathbf{K}$. The surface is given as a level set of the function f, so its normal is

$$\nabla f(x,y) = \frac{x}{2}\mathbf{I} + 2y\mathbf{J} + \frac{2z}{9}\mathbf{K} \ .$$

The places on Σ where the tangent plane is parallel to the given plane are those values of (x,y) where $\nabla f(x,y)$ is colinear with $\mathbf{N}$. These are the solutions of the system of equations:

$$x = 4\lambda, \quad y = \frac{\lambda}{2}, \quad z = -\frac{9\lambda}{2}, \quad \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1 \ .$$

Putting the expressions in λ given by the first three equations into the fourth, we can solve for λ, getting

$$\lambda = \pm\frac{2}{\sqrt{26}} \ .$$

Thus there are two solutions to the problem:

$$\left(\frac{8}{\sqrt{26}}, \frac{1}{\sqrt{26}}, z = -\frac{9}{\sqrt{26}}\right) \ , \quad \left(-\frac{8}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, z = \frac{9}{\sqrt{26}}\right) \ .$$

b) What constrained optimization problem is solved by part a)?

Solution. Find the maximum and minimum of $2x + y - z$ on the surface Σ.

10. Find the volume of the tetrahedron in the first octant bounded by the plane

$$\frac{x}{5} + \frac{y}{3} + \frac{z}{2} = 1 \ .$$

Solution. Draw the picture to see that we can represent the region by the inequalities

$$0 \leq x \leq 5 \ , \quad 0 \leq y \leq 3(1 - \frac{x}{5}) \ , \quad 0 \leq z \leq 2(1 - \frac{x}{5} - \frac{y}{3}) \ .$$

So the volume is given by the iterated integral

$$\int_0^5 \int_0^{3(1-\frac{x}{5})} \int_0^{2(1-\frac{x}{5}-\frac{y}{3})} dzdydx = \frac{15}{2} \ .$$

11. a) Find the volume of the solid in the first quadrant which lies over the triangle with vertices (0,0), (1,0), (1,3) and under the plane $z = 2x + 3y + 1$.

Solution. The solid is that under the given plane and lying over the triangle $T : 0 \leq x \leq 1,\ 0 \leq y \leq 3x$. Its volume is

$$Volume = \int\int_T zdxdy = \int_0^1 \int_0^{3x} (2x + 3y + 1)dydx \ .$$

The inner integral is

$$2xy + \frac{3y}{2} + y\Big|_0^{3x} = \frac{21}{2}x^2 + 3x \ .$$

Thus

$$Volume = \int_0^1 (\frac{21}{2}x^2 + 3x)dx = \frac{21}{6} + \frac{3}{2} = 5 \ .$$

b) Find the area of that segment of the plane.

Solution. The element of surface area is $dS = \sqrt{1 + z_x^2 + z_y^2}dxdy = \sqrt{1 + 2^2 + 3^2}dxdy = \sqrt{14}dxdy$. Thus the area of the triangle on the surface is $\sqrt{14}$ times the area of the triangle, so is $3\sqrt{14}/2$.

12. Find the area of the region in the first quadrant bounded by the parabolas

$$y^2 - x = 1, \quad y^2 - x = 0, \quad y^2 + x = 5, \quad y^2 + x = 4 \ .$$

Solution. Make the change of variable: $u = y^2 - x,\ v = y^2 + x$. Then in the uv-plane the region is described by $0 \le u \le 1, 4 \le v \le 5$. We need to calculate the Jacobian; for that we solve for x and y in terms of u and v:

$$x = \frac{v-u}{2} \quad y = \frac{(u+v)^{1/2}}{\sqrt{2}} \ .$$

Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{(u+v)^{-1/2}}{2\sqrt{2}} & \frac{(u+v)^{-1/2}}{2\sqrt{2}} \end{pmatrix} = -\frac{(u+v)^{-1/2}}{2\sqrt{2}} \ .$$

The area then is the integral

$$Area = \frac{1}{2\sqrt{2}}\int_0^1\int_4^5 (u+v)^{-1/2}dvdu \ .$$

The inner integral is $2[(u+5)^{1/2} - (u+4)^{1/2}]$, so

$$Area = \frac{1}{\sqrt{2}}\int_0^1 [(u+5)^{1/2} - (u+4)^{1/2}]du = \frac{\sqrt{2}}{3}(6^{3/2} + 4^{3/2} - 5^{3/2}) \ .$$

13. Find the mass of a lamina over the domain in the plane $D:\ 0 \le y \le x(1-x)$, if the density function is $\delta(x,y) = 1 + x + y$.

Solution.

$$Mass = \int\int_D \delta dA = \int_0^1\int_0^{x(1-x)} (1+x+y)dydx \ .$$

The inner integral is

$$y + xy + \frac{y^2}{2}\Big|_0^{x(*1-x)} = \frac{x^4}{2} - 2x^3 + \frac{x^2}{2} + x \ .$$

Thus

$$Mass = \frac{1}{10} - \frac{2}{4} + \frac{1}{6} + \frac{1}{2} = \frac{4}{15} \ .$$

14. Find the center of mass of the piece of the unit sphere in the first octant:

$$x^2 + y^2 + z^2 \le 1\ , \qquad x \ge 0, \quad y \ge 0, \quad z \ge 0 \ .$$

Solution. The volume of the sphere of radius 1 is $4\pi/3$; the piece we're looking at is (1/8)thof that so $Mass = \pi/6$. Now, using spherical coordinates

$$Mom_{x=0} = \int\int\int_R xdV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\sin\phi\cos\theta\rho^2\sin\phi d\rho d\theta d\phi$$

$$= \int_0^{\pi/2} \cos\theta d\theta \int_0^{\pi/2} \sin^2\phi d\phi \int_0^1 \rho^3 d\rho = \frac{\pi}{16} \ .$$

Thus the x-coordinate of the center of mass is

$$\bar{x} = \frac{Mom_{x=0}}{Mass} = \frac{\frac{\pi}{16}}{\frac{\pi}{6}} = \frac{3}{8} \ .$$

15. Let

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \ .$$

Find a) ∇f, b) curl ∇f, c) div ∇f, d) $\nabla(\text{div } \nabla f)$.

Solution.

a)

$$\nabla f = (\frac{1}{y} - \frac{z}{x^2})\mathbf{I} + (\frac{1}{z} - \frac{x}{y^2})\mathbf{J} + (\frac{1}{x} - \frac{y}{z^2})\mathbf{K}$$

b)

$$\text{curl } \nabla f = 0$$

c)

$$\text{div } \nabla f = \frac{2z}{x^3} + \frac{2x}{y^3} + \frac{2y}{z^3}$$

d)

$$\nabla(\text{div } \nabla f) = (\frac{-6z}{x^4} + \frac{2}{y^3})\mathbf{I} + (\frac{-6x}{y^4} + \frac{2}{z^3})\mathbf{J} + (\frac{-6y}{z^4} + \frac{2}{x^3})\mathbf{K}$$

e)

$$\nabla \times \nabla(div \nabla f) = 0 \quad .$$

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16. Let $\mathbf{F} = (y + 2xz)\mathbf{I} + (x + z^2 + 1)\mathbf{J} + (2yz + x^2\mathbf{K}$. Find a function f such that $\mathbf{F} = \nabla f$.

Solution. We want to solve the equations

$$\frac{\partial f}{\partial x} = y + 2xz \ , \quad \frac{\partial f}{\partial y} = x + z^2 + 1 \ , \quad \frac{\partial f}{\partial z} = 2yz + x^2 \ .$$

The general solution of the first equation is $f(x,y,z) = xy + x^2z + \phi(y,z)$. Substitute this in the second equation to get

$$\frac{\partial f}{\partial y} = x + \frac{\partial \phi}{\partial y} = x + z^2 + 1 \ ,$$

leading to the equation for ϕ:

$$\frac{\partial \phi}{\partial y} = z^2 + 1 \ .$$

This has the solution $\phi(y,z) = z^2 y + y + \psi(z)$. This now gives this form for f:

$$f(x,y,z) = xy + x^2 z + z^2 y + y + \psi(z) \ .$$

Substitute that in the last equation to get

$$\frac{\partial f}{\partial z} = x^2 + 2zy + \psi'(z) = 2yz + x^2 \ ,$$

so that we must have $\psi'(z) = 0$, or $\psi(z) = C$. Thus the answer is

$$f(x,y,z) = xy + x^2 z + z^2 y + y + C \ .$$

17. Let C be the curve in space given parametrically by the equations

$$x = t^2 - 3t + 5 \ , \quad y = (t^3 - 2)^2 \ , \quad z = t^4 + t^3 - t^2, 0 \le t \le 1 \ ,$$

and $\mathbf{F}$ the vector field

$$\mathbf{F}(x,y,z) = x\mathbf{I} + z\mathbf{J} + y\mathbf{K} \ .$$

What is $\int_C \mathbf{F} \cdot d\mathbf{X}$?

Solution. Before doing the hair-raising direct calculation, note that $\mathbf{F} = \nabla(x^2+y^2+z^2)/2$. Thus we need only evaluate this function at the endpoints, which are (5,4,0) (for $t = 0$) and (3,1,1) (for $t = 1$). Thus

$$\int_C \mathbf{F} \cdot d\mathbf{X} = \frac{x^2+y^2+z^2}{2}\Big|_{(5,4,0)}^{(3,1,1)} = 15 \ .$$

18. Let C be the curve given in polar coordinates by $r = 1 + \cos\theta, \ 0 \le \theta \le 2\pi$. Calculate $\int_C x dy$.

Solution. Parametrize the curve by $x = (1+\cos\theta)\cos\theta, \ y = (1+\cos\theta)\sin\theta, \ 0 \le \theta \le 2\pi$, so that

$$\int_C x dy = \int_0^{2\pi} (1+\cos\theta)\cos\theta d((1+\cos\theta)\sin\theta)$$

$$= \int_0^{2\pi} (1+\cos\theta)\cos\theta(-\sin^2\theta + (1+\cos\theta)\cos\theta)d\theta = \frac{3}{2}\pi \ .$$

Instead of computing that awful integral, we could note that, by Green's theorem, the desired integral is the area of the cardiod D bounded by C, so

$$\int_C x dy = Area(D) = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1 + 2\cos\theta + \frac{1+\cos(2\theta)}{2})d\theta = \frac{3}{2}\pi\ .$$

19. Let C be the part of the curve $y = x^2(24 - x)$ which lies in the first quadrant. Consider it directed from the point (0,0) to the point (24,0). Calculate

$$\int_C (y+1)dx - xdy\ .$$

Solution. We can parametrize this curve by $y = 24x^2 - x^3,\ 0 \le x \le 24$; in which case $dy = (48x - 3x^2)dx$ and we get

$$\int_C (y+1)dx - xdy = \int_0^{24} (24x^2 - x^3 + 1)dx - x(48x - 3x^2)dx$$

$$= \int_0^{24} (-24x^2 - 4x^3 + 1)dx = -442344\ .$$