Calculus III
Practice Exam 3, Answers

1. Find the volume under the plane $z = x + 2y + 1$ over the triangle bounded by the lines $y = 0,\ x = 1,\ y = 2x$.

Answer. We view this as the region lying over the triangle T and under the plane $z = x + 2y + 1$. T is the type 1 domain $0 \le x \le 1,\ 0 \le y \le 2x$. Thus

$$Volume = \int\int_T z dx dy = \int_0^1 [\int_0^{2x} (x + 2y + 1) dy] dx \ .$$

The inner integral is

$$\int_0^{2x} (x + 2y + 1) dy = (xy + y^2 + y)\Big|_0^{2x} = 6x^2 + 2x \ .$$

Thus

$$Volume = \int_0^1 (6x^2 + 2x) dx = \frac{6}{3} + 1 = 3 \ .$$

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2. Let R be the region in the plane bounded by the curves $x = y^2$, $x = 3 - 2y^2$. Calculate

$$I = \int\int_R (y^2 - x) dx \ .$$

Answer. This is the region between two parabolas with axis the x-axis. Integrate first with respect to x along curves y = constant from y^2 to $3 - 2y^2$. To find the range in y find the point of intersection of the parabolas: $y^2 = 3 - 2y^2$; solutions $y = \pm 1$. Thus

$$I = \int_{-1}^1 \Big[\int_{y^2}^{3-y^2} (y^2 - x) dx\Big] dy \ .$$

The inner integral is

$$\Big[y^2 x - \frac{x^2}{2}\Big]\Big|_{y^2}^{3-y^2} = -2y^4 + 6y^2 - \frac{9}{2} \ .$$

Integrating over y, we obtain

$$I = \int_{-1}^1 (-2y^4 + 6y^2 - \frac{9}{2}) dy = 2[-\frac{2}{5} - +2 - \frac{9}{2}] = -\frac{29}{5} \ .$$

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3. Let R be the region in the first quadrant bounded by the curves $y = x$ and $y = x^3$. What are the coordinates of its centroid?

Answer. R is the type 1 region $0 \le x \le 1,\ x^3 \le y \le x$. Thus

$$Area = \int_0^1 \int_{x^3}^x dy dx = \int_0^1 (x - x^3) dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \ ,$$

$$Mom_{\{x=0\}} = \int_0^1 \int_{x^3}^x x dy dx = \int_0^1 (x^2 - x^4) dx = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} \ ,$$

$$Mom_{\{y=0\}} = \int_0^1 \int_{x^3}^x y dy dx = \int_0^1 (\frac{y^2}{2}\Big|_{x^3}^x) dx = \frac{1}{2}\int_0^1 (x^2 - x^6) dx = \frac{1}{2}(\frac{1}{3} - \frac{1}{7}) = \frac{2}{21} \ .$$

The centroid is at (8/15,8/21).

4. What is the mass of the lamina bounded by the curves $y = 3x$ and $y = 6x - x^2$ where the density function is $\delta(x,y) = xy$?

Answer. This is the type 1 region $0 \le x \le 3, 3x \le y \le 6x - x^2$. Thus

$$Mass = \int\int_R \delta dA = \int_0^3 [\int_{3x}^{6x-x^2} xydy]dx \ .$$

The inner integral is

$$x\int_{3x}^{6x-x^2} ydy = \frac{x}{2}((6x-x^2)^2 - (3x)^2) = \frac{x^3}{2}(x^2 - 12x + 27) \ .$$

Thus

$$Mass = \frac{1}{2}\int_0^3 (x^5 - 12x^4 + 27x^3)dx = \frac{9 \cdot 81}{2}(\frac{1}{6} - \frac{4}{5} + \frac{3}{4}) = \frac{729}{2}\frac{7}{60} = 42.525$$

5. As (u,v) runs through the region $u^2 + v^2 \le 1$, the vector function

$$\mathbf{X}(u,v) = (u^2+v^2)\mathbf{I} + (u^2-v^2)\mathbf{J} + uv\mathbf{K}$$

describes a surface S in three space. Write down the double integral which must be calculated to find the surface area of S.

Answer.

$$\mathbf{X}_u = 2u\mathbf{I} + 2u\mathbf{J} + v\mathbf{K}, \qquad \mathbf{X}_v = 2v\mathbf{I} - 2v\mathbf{J} + u\mathbf{K} \ ,$$

$$\mathbf{X}_u \times \mathbf{X}_v = (2v^2 + 2u^2)\mathbf{I} + (2v^2 - 2u^2)\mathbf{J} - 4uv\mathbf{K} \ ,$$

so, when the smoke clears we find

$$|\mathbf{X}_u \times \mathbf{X}_v| = 2\sqrt{2}|u^2 - v^2| \ .$$

Thus

$$S = 2\sqrt{2}\int\int_R |u^2 - v^2|dudv \ ,$$

where R is the unit disc. We now switch to polar coordinates: $u^2 - v^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$, so

$$S = 2\sqrt{2}\int_0^{2\pi}\int_0^1 |\cos 2\theta| r^3 dr d\theta = \frac{\sqrt{2}}{2}\int_0^{2\pi} |\cos 2\theta| d\theta = 4\sqrt{2}\int_0^{\pi/4} \cos 2\theta d\theta =$$

$$4\sqrt{2} \cdot \frac{1}{2}\frac{\sqrt{2}}{2} = 2$$

6. Find the volume of the region bounded below by the surface $z = 4x^2 + 25y^2$, and above by the plane $z = 100$.

Answer. This can be viewed as the volume of the region between the surfaces $z = 100$ and $z = 4x^2 + 25y^2$ which lies above the ellipse $E : 4x^2 + 25y^2 \le 100$:

$$Volume = \int\int_E (100 - 4x^2 - 25y^2)dxdy \ .$$

Let $u = 2x, \ v = 5y$. Then E corresponds to the disc $D : u^2 + v^2 \le 100$ and $100 - 4x^2 - 25y^2 = 100 - u^2 - v^2$. Since $x = u/2, \ y = v/5$, the Jacobian of the transformation is $1/10$.This gives us

$$Volume = \int\int_D f(x(u,v), y(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv = \frac{1}{10}\int\int_D (100 - u^2 - v^2)dudv \ .$$

Now, moving to polar coordinates in u, v space, this becomes

$$\frac{1}{10}\int_0^{2\pi}\int_0^{10}(100-r^2)rdrd\theta = \frac{\pi}{5}\int_0^{100}(100-r^2)dr = 500\pi \ .$$

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7. Find the centroid of the region under the cone $z^2 = x^2+y^2$ lying over the disc $x^2+y^2 \leq 9$.

Answer. In cylindrical coordinates, this region is describe by the inequalities $r \leq 3$, $0 \leq z \leq r$. Thus,

$$Volume = \int_0^{2\pi}\int_0^3\int_0^r rdzdrd\theta = 2\pi\int_0^3 r^2dr = 18\pi \ .$$

Because of the symmetry about the z-axis, the centroid lies on the z-axis. To find $\bar{z}$, we calculate the moment

$$Mom_{\{z=0\}} = \int_0^{2\pi}\int_0^3\int_0^r rzdzdrd\theta = 2\pi\int_0^3 \frac{r^3}{2}dr = \frac{81\pi}{2} \ .$$

Thus $\bar{z} = (81\pi/4)/(18\pi) = 9/8$.

8. Find the volume inside the hyperboloid $x^2+y^2-z^2 = 1$, for $0 \leq z \leq 2$.

Answer. In cylindrical coordinates, the region is described by the inequalities $0 \leq z \leq 2$, $0 \leq r \leq \sqrt{1+z^2}$. Thus

$$Volume = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{1+z^2}} rdrdzd\theta \ .$$

The inner integral is

$$\int_0^{\sqrt{1+z^2}} rdr = \frac{1+z^2}{2} \quad \text{so} \quad Volume = 2\pi\int_0^2 \frac{1+z^2}{2}dz = \frac{14\pi}{3} \ .$$

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9. Find the surface area of the piece of the paraboloid $z = x^2+y^2$ lying between the planes $z = 0, \ z = 2$.

Answer. We start with the equation $dS = \sqrt{1+z_x^2+z_y^2}dxdy$. Since $z_x = 2x, \ z_y = 2y$, this gives us $dS = \sqrt{1+4x^2+4y^2}dxdy = \sqrt{1+4r^2}rdrd\theta$, where we have switched to polar coordinates because of the circular symmetry. Then the area is the integral of dS over the disc $r \leq \sqrt{2}$:

$$Surface\ Area = \int_0^{2\pi}\int_0^{\sqrt{2}}(1+r^2)^{1/2}rdrd\theta = 2\pi\int_1^3 u^{1/2}du = \frac{2\pi}{3}(3\sqrt{3}-1) \ ,$$

using the substitution $u = 1+r^2$.

10. The part R of the sphere of radius 1 centered at the origin which lies in the first octant is filled with a material whose density function is $\delta(x,y,z) = z^2+xy$. Find the mass of this object.

Answer. The mass is the triple integral over R of δdV. Switching to spherical coordinates, R is described by the inequalities $0 \leq \theta \leq \pi/2, \ 0 \leq \phi \leq \pi/2, 0 \leq \rho \leq 1$, and $\delta = \rho^2(\cos^2\phi + \sin^2\phi\cos\theta\sin\theta)$.

$$Mass = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho^2(\cos^2\phi+\sin^2\phi\cos\theta\sin\theta)\rho^2\sin\phi d\phi d\theta \ .$$

Now, the integral with respect to ρ is 1/5. To calculate the integral with respect to ϕ, we use

$$\int_0^{\pi/2}\cos^2\phi\sin\phi d\phi = \frac{1}{3} \qquad \int_0^{\pi/2}\sin^2\phi d\phi = \frac{2}{3} \ .$$

Thus

$$Mass = \frac{1}{15}\int_0^{\pi/2}(1+2\cos\theta\sin\theta)d\theta = \frac{1}{15}(\frac{\pi}{2}+1) \ .$$