Calculus III
Practice Exam 2, Answers

1. A conic in the plane is given by the equation

$$x^2 - 2xy + y^2 + 2x - y = 0 \ .$$

a) What conic is it?

Answer. Since $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$, this is a parabola. We can verify that by completing the square to obtain: $(x-y)^2 + 2x - y = 0$, which is the parabola $u^2 + v = 0$ in the coordinates $u = x - y, v = 2x - y$.

b) At what angle to the x-axis are the axes of the conic?

Answer. If θ is that angle, then $\tan(2\theta) = B/(A-C)$. Since $A = C$, $\theta = \pi/4$.

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2. A conic in the plane is given by the equation $5x^2 - xy + y^2 = 50$.

a) What conic is it?

Answer. Since $B^2 - 4AC = (-1)^2 - 4(5)(1) = -16 < 0$, this is an ellipse.

b) At what angle to the x-axis are the axes of the conic?

Answer. If θ is that angle, then $\tan(2\theta) = B/(A-C) = -1/(5-1) = -1/4$. We find $\theta = .0390\pi$ radians.

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3. A parabola has its vertex at the origin, and its focus at the point (3,4). Give the equation of the parabola.

Answer. $\mathbf{F} = 3\mathbf{I} + 4\mathbf{J}$ is the vector from the vertex to the focus of the parabola. Thus **F** lies in the direction of the axis of the parabola. Let **L** be the unit vector in this direction, and let u, v be the coordinates of a point relative to the axis base $\mathbf{L}$, $\mathbf{L}^\perp$. in these coordinates, the parabola is in standard position, so its equation is $v^2 = 4pu$, where p is the distance between focus and vertex; thus $p = |\mathbf{F}| = \sqrt{3^2 + 4^2} = 5$, and the equation is $v^2 = 20u$. Now, the point represented by the vector $\mathbf{X} = x\mathbf{I} + y\mathbf{J} = u\mathbf{L} + v\mathbf{L}^\perp$ has coordinates $u = \mathbf{X} \cdot \mathbf{L}$, $v = \mathbf{X} \cdot \mathbf{L}^\perp$. Calculating, with

$$\mathbf{L} = \frac{3}{5}\mathbf{I} + \frac{4}{5}\mathbf{J} \ , \quad \mathbf{L}^\perp = -\frac{4}{5}\mathbf{I} + \frac{3}{5}\mathbf{J} \ ,$$

we get

$$u = \mathbf{X} \cdot \mathbf{L} = \frac{3}{5}x + \frac{4}{5}y \ , \quad v = \mathbf{X} \cdot \mathbf{L}^\perp = -\frac{4}{5}x + \frac{3}{5}y \ .$$

Making this substitution in the equation $v^2 = 20u$, we get

$$\frac{(-4x+3y)^2}{25} = 20\frac{3x+4y}{5} \ ,$$

which simplifies to $16x^2 - 24xy + 9y^2 - 300x - 400y = 0$.

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4. Let $f(x,y) = 3x^2y + 3xy$.

Answer. a) $\nabla f = (6xy + 3y)\mathbf{I} + (3x^2 + 3x)\mathbf{J}$.?

b) What is the direction of maximum increase of f at the point (1,2)?

Answer. This is the direction of the gradient. At the point (1,2) we have $\nabla f = 18\mathbf{I} + 6\mathbf{J}$, so the direction we seek is $\mathbf{U} = (3\mathbf{I} + \mathbf{J})/\sqrt{10}$.

c) What are the critical points of f? What kind of critical points are they?

Answer. We must solve $6xy + 3y = 0, 3x^2 + 3x = 0$. From the second equation $x = 0$ or $x = -1$. From the first, when $x = 0, y = 0$, and when $x = -1, y = 2$. The critical points are (0,0),(-1,2). The second partials are

$$f_{xx} = 6y\ , \quad f_{xy} = 6x + 3\ , \quad f_{yy} = 0\ .$$

At both points then, $AC - B^2 < 0$, so they are both saddle points.

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5. Let

$$f(x,y) = \frac{1}{x} + \frac{1}{y}\ .$$

a) What is the tangent line to the curve $f(x,y) = 5/6$ at the point (2,3)?

Answer. Differentiating, we get $x^{-2}dx + y^{-2}dy = 0$. At the point this evaluates to $dx/4 + dy/9 = 0$. Replacing the differentials by the increments, we have the equation of the tangent line:

$$\frac{x-2}{4} + \frac{y-3}{9} = 0 \quad \text{or} \quad \frac{x}{4} + \frac{y}{9} = \frac{5}{6}\ .$$

b) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point (2,3,5/6).

Answer. The surface is given by the equation $z - x^{-1} - y^{-1} = 5/6$. Taking differentials we have $dz + x^{-2}dx + y^{-2}dy = 0$. At the point (2,3,5/6) we have $dz + dx/4 + dy/9 = 0$. Replacing the differentials by the increments gives the equation of the tangent plane:

$$(z - \frac{5}{6}) + \frac{(x-2)}{4} + \frac{y-3}{9} = 0\ , \quad \text{or} \quad \frac{x}{4} + \frac{y}{9} + z = \frac{5}{3}\ .$$

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6. Let

$$f(x,y,z) = \frac{1}{xy} + \frac{1}{yz}\ .$$

What is the equation of the tangent plane to the level surface $f(x,y,z) = 1$ at the point (1,2,1)?

Answer. We have

$$\nabla f = -\frac{1}{x^2y}\mathbf{I} - (\frac{1}{xy^2} + \frac{1}{zy^2})\mathbf{J} - \frac{1}{z^2y}\mathbf{K}\ .$$

The value at (1,2,1) is the normal to the tangent plane, so we have $\mathbf{N} = -.5(\mathbf{I} + \mathbf{J} + \mathbf{K})$. Now, $\mathbf{X}_0 = \mathbf{I} + 2\mathbf{J} + \mathbf{K}$ is a point on the plane, so the equation of the plane is

$$\mathbf{N} \cdot \mathbf{X} = \mathbf{N} \cdot \mathbf{X_0}\ , \quad \text{or} \quad x + y + z = 4\ .$$

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7. Let $w = x\sqrt{y} + y\sqrt{z}$, and let γ be the curve $x = -t,\ y = t^2,\ z = 1 + t$, for $t > 0$. What is dw/dt at $t = 1$?

Answer. We calculate

$$\nabla w = \sqrt{y}\mathbf{I} + (\frac{x}{2\sqrt{y}} + \sqrt{z})\mathbf{J} + \frac{y}{2\sqrt{z}}\mathbf{K} \,, \qquad \frac{d\mathbf{X}}{dt} = -\mathbf{I} + 2t\mathbf{J} + \mathbf{K} \,.$$

At $t = 1$, we have $x = -1, y = 1, z = 2$, and the values

$$\nabla w = \mathbf{I} + (\sqrt{2} - (1/2))\mathbf{J} + \frac{1}{2\sqrt{2}}\mathbf{K} \,, \quad \text{and} \quad \frac{d\mathbf{X}}{dt} = -\mathbf{I} + 2\mathbf{J} + \mathbf{K} \,.$$

Thus

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{X}}{dt} = -2 + 2\sqrt{2} + \frac{1}{2\sqrt{2}} \,.$$

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8. Let

$$f(x,y) = x^3 y + \frac{1}{2} y^2 x + y x^2 \,.$$

Find all saddle points of the surface $z = f(x,y)$.

Answer. We calculate $f_x = 3x^2 y + (1/2)y^2 + 2xy$, $f_y = x^3 + xy + x^2$. Setting both equal to zero, the second equation tells us that either $x = 0$ or $y = -x - x^2$. In the first case, the first equation gives $y = 0$, so (0,0) is a critical point. In the second case we can substitute $y = -x - x^2$ in the first equation and solve for x. There's some algebra to do, but it leads to the equation $5x^2 + 8x + 3 = 0$ (we can factor out an x^2, since we are in the case $x \neq 0$. This has the solutions $x = -1$ or $x =, .6$, with the corresponding y-values $0, .24$.

Thus the critical points are (0,0), (-1,0), (-.6,.24). To find the type of critical point, we calculate

$$f_{xx} = 6xy + 2y \,, \quad f_{xy} = 3x^2 + y + 2x \,, \quad f_{yy} = x \,.$$

Evaluating at the points, we get no information at (0,0) (all second partial derivatives are zero), and that the other two points are saddle points.

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9. Find the point on the curve $2(x-1)^2 + 3y^2 = 22$ which is closest to the origin.

Answer. Here the objective function is $f(x,y) = x^2 + y^2$ and the constraint is $g(x,y) = 2(x-1)^2 + 3y^2 = 22$. We calculate

$$\nabla f = 2x\mathbf{I} + 2y\mathbf{J} \,, \quad \nabla g = 4(x-1)\mathbf{I} + 6y\mathbf{J} \,,$$

so Lagrange's equations are

$$x = 2\lambda(x-1) \,, \quad y = 3\lambda y \,, \quad 2(x-1)^2 + 3y^2 = 22 \,.$$

From the second equation either $y = 0$ or $\lambda = 1/3$.
Case $y = 0$. The third equation gives $x = 1 \pm \sqrt{2}$, and the possible values of f at these points are $3 \pm 2\sqrt{2}$.
Case $\lambda = 1/3$. The first equation becomes $x = (2/3)(x-1)$, which has the solution $x = -2$. Put that in the last equation, and solve for y to get $y = \pm 2/\sqrt{3}$. The corresponding values of f are both $4 + 4/3$. Thus the smallest of these possible values is $3 - \sqrt{2}$, which is taken at the point $(1 - \sqrt{2}, 0)$.

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10. A rectangular box of maximum volume is to be constructed, with sides parallel to the coordinate planes, one corner at the origin and the diagonally opposite corner on the plane $2x + 3y + z = 1$. What are the dimensions of the box?

Answer. Here the objective function is $V = xyz$, and the constraint is $g(x,y,z) = 2x+3y+z = 1$. We calculate:

$$\nabla V = yz\mathbf{I} + xz\mathbf{J} + xy\mathbf{K}\ , \quad \nabla g = 2\mathbf{I} + 3\mathbf{J} + \mathbf{K}\ .$$

The Lagrange equations are

$$yz = 2\lambda\ , \quad xz = 3\lambda, \quad xy = \lambda\ , \quad 2x+3y+z = 1\ .$$

If we multiply the first equation by x, the second by y, and the third by z, we find

$$xyz = 2x\lambda = 3y\lambda = z\lambda\ .$$

Since $\lambda \neq 0$ (for if so, some coordinate is zero, and thus $V = 0$, which is certainly not the maximum), this gives us $2x = z,\ 3y = z$. Put that in the constraint equation and solve for z: $2x+3y+z = 3z = 1$, so $z = 1/3$ and thus $x = 1/6$ and $y = 1/9$. Thus the maximum volume is $V = 1/162$m taken at the point (1/6,1/9,1/3).