Calculus III
Practice Exam 1, Answers

1. $\mathbf{V} = 4\mathbf{I} + 7\mathbf{J},\ \mathbf{W} = -\mathbf{I} + 5\mathbf{J}$ are two vectors in the plane.

a) Find the angle between $\mathbf{V}$ and $\mathbf{W}$.

Answer. Let α be the angle. Then

$$\cos\alpha = \frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{V}||\mathbf{W}|} = \frac{4(-1)+7(5)}{\sqrt{65}\sqrt{26}} = \frac{31}{\sqrt{1690}} = .7508\ ,$$

so $\alpha = \arccos(.7508) = .7165$ radians.

b) Find the angle between $\mathbf{V}$ and $\mathbf{W}^{\perp}$.

Answer. $\mathbf{W}^{\perp} = -5\mathbf{I} - \mathbf{J}$, so, if β is the desired angle,

$$\cos\beta = \frac{\mathbf{V}\cdot\mathbf{W}^{\perp}}{|\mathbf{V}||\mathbf{W}|} = \frac{4(-5)+7(-1)}{\sqrt{-65}\sqrt{26}} = \frac{-27}{\sqrt{1690}} = -.6568\ ,$$

so $\beta = \arccos(-.6568) = 2.2873$ radians. Of course, since $\mathbf{W}$ is counterclockwise to $\mathbf{W}$, $\mathbf{W}^{\perp}$ is another $\pi/2$ counterclockwise to $\mathbf{W}$, so

$$\beta = \frac{\pi}{2} + \alpha = 1.5708 + .7165 = 2.2873\ .$$

c) Find the area of the parallelogram spanned by $\mathbf{W}$ and $\mathbf{W}^{\perp}$.

Answer. This is $\det(\mathbf{W},\mathbf{W}^{\perp}) = 26$. This can also be computed as $|\mathbf{W}|^2$, since the parallelogram in question is a square of side length $|\mathbf{W}|$.

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2. Find the distance of the point (2,-2) from the line given by the equation $x + 2y = 8$.

Answer. Let Q be the given point. We need a point on the line; $P(0,4)$ will do. The distance is the length of the projection of $\overrightarrow{PQ} = 2\mathbf{I} - 6\mathbf{J}$ in the direction normal to the line. The vector formed of the coefficeints of the equation of the line: $\mathbf{N} = \mathbf{I} + 2\mathbf{J}$ is normal to the line. Thus the distance is

$$\frac{|\overrightarrow{PQ}\cdot\mathbf{N}|}{|\mathbf{N}|} = \frac{|2-12|}{\sqrt{5}} = 2\sqrt{5}$$

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3. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = t\mathbf{I} - t^3\mathbf{J}$$

Find the velocity, speed, acceleration, tangent and normal vectors, and normal acceleration of the particle at any time t.

Answer. Differentiate $\mathbf{X}(t)$:

$$\mathbf{V} = \mathbf{I} - 3t^2\mathbf{J}\ ,\quad \mathbf{A} = 6t\mathbf{J}\ .$$

$$\frac{ds}{dt} = |\mathbf{V}| = \sqrt{1+9t^4},\quad \mathbf{T} = \frac{\mathbf{I} - 3t^2\mathbf{J}}{\sqrt{1+9t^4}},\quad \mathbf{N} = \pm\frac{3t^2\mathbf{I} + \mathbf{J}}{\sqrt{1+9t^4}}\ .$$

$$a_N = \mathbf{A}\cdot\mathbf{N} = \pm\frac{6t\mathbf{J}\cdot(3t^2\mathbf{I}+\mathbf{J})}{\sqrt{1+9t^4}} = \frac{\pm 6t}{\sqrt{1+9t^4}}\ .$$

Since a_N must be positive, $a_N = 6|t|/\sqrt{1+9t^4}$.

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4. Find the symmetric equation of the line through the point (2,-1,3) which is perpendicular to the vectors $\mathbf{I}-2\mathbf{J}+3\mathbf{K}$ and $3\mathbf{I}-2\mathbf{J}+\mathbf{K}$. **Answer**. The direction vector for the line is the cross product of the given

vectors:

$$\mathbf{L} = (\mathbf{I}-2\mathbf{J}+3\mathbf{K})\times(3\mathbf{I}-2\mathbf{J}+\mathbf{K}) = 4\mathbf{I}+8\mathbf{J}+4\mathbf{K}$$

. Thus the symmetric equations are

$$\frac{x-2}{4} = \frac{y+1}{8} = \frac{z-3}{4}\ , \quad \text{or} \quad x-2 = \frac{y+1}{2} = z-3\ .$$

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5. Find the equation of the plane through the origin which is normal to the line given parametrically by

$$\mathbf{X} = (3\mathbf{I}+2\mathbf{J}-\mathbf{K})+t(-\mathbf{I}+\mathbf{J}+2\mathbf{K})\ .$$

Answer. Since $-\mathbf{I}+\mathbf{J}+2\mathbf{K}$ is the direction vector of the line, it is the normal to the plane. Since the plane goes through the origin, its equation is $-x+y+2z=0$.

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6. Find a vector normal to the plane through $(0,0,0),\ (1,0,-1),\ (0,1,1)$.

Answer. Since the origin is on this plane, the vectors from the origin to the second and third points,$\mathbf{I}-\mathbf{K},\ \mathbf{J}+\mathbf{K}$, are on the plane, so the normal $\mathbf{N}$ is their cross product:

$$\mathbf{N} = (\mathbf{I}-\mathbf{K})\times(\mathbf{J}+\mathbf{K}) = \mathbf{I}-\mathbf{J}+\mathbf{K}\ .$$

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7. Consider two different, but parallel planes given by the equations

$$\Pi_1\ :\quad (\mathbf{X}-\mathbf{X_1})\cdot\mathbf{N} = 0\ , \qquad \Pi_2\ :\quad (\mathbf{X}-\mathbf{X_2})\cdot\mathbf{N} = 0\ .$$

Express the distance between the planes as a function of $\mathbf{X_1},\ \mathbf{X_2},\mathbf{N}$. **Answer**. Since the planes are parallel, the length of any line segment between the planes and perpendicular to both planes will give this distance. Thus, we need only find the length of the projection of $\mathbf{X_2}-\mathbf{X_1}$ along the normal $\mathbf{N}$:

$$d = \frac{|(\mathbf{X_2}-\mathbf{X_1})\cdot\mathbf{N}}{|\mathbf{N}|}\ .$$

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8. Find the distance of the point $(3,2,1)$ from the line whose symmetric equations are

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-1}{-2}$$

Answer. The direction vector of the line is $\mathbf{L} = 3\mathbf{I}+4\mathbf{J}-2\mathbf{K}$, and $P(2,-1,1)$ is a point on the line. Consider the parallelogram determined by the vectors $\mathbf{L}$ and $\overrightarrow{PQ}$, where Q is the given point. Its area is $|\mathbf{L}\times\overrightarrow{PQ}|$. But this area is also the product of the length of one side and the distance to the opposing side; this first is $|\mathbf{L}|$, and the second is d, the desired distance. Thus $|\mathbf{L} \times \overrightarrow{PQ}| = d|\mathbf{L}|$. So, we must compute:

$$\mathbf{L} \times \overrightarrow{PQ} = (3\mathbf{I} + 4\mathbf{J} - 2\mathbf{K}) \times (\mathbf{I} + 3\mathbf{J}) = 6\mathbf{I} - 2\mathbf{J} + 5\mathbf{K} .$$

This length is $\sqrt{65}$, and the length of $\mathbf{L}$ is $\sqrt{29}$, so $d = \sqrt{65/29} = 1.497$.

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9. A particle moves in space according to the formula $\mathbf{X}(t) = \mathbf{I} + t\mathbf{J} - t^2\mathbf{K}$. Find the tangential and normal accelerations as functions of t.

Answer. Differentiate:

$$\mathbf{V}(t) = \mathbf{J} - 2t\mathbf{K} , \quad \mathbf{A}(t) = -2\mathbf{K} .$$

Thus

$$\mathbf{T} = \frac{\mathbf{J} - 2t\mathbf{K}}{\sqrt{1+4t^2}} , \quad a_T = \mathbf{A} \cdot \mathbf{T} = \frac{4t}{\sqrt{1+4t^2}} .$$

Since $\mathbf{V}$ and $\mathbf{A}$ are always in the $\mathbf{J},\mathbf{K}$-plane, we can take

$$\mathbf{N} = \mathbf{T}^{\perp} = \frac{-2t\mathbf{J} - \mathbf{K}}{\sqrt{1+4t^2}} , \quad a_N = \mathbf{A} \cdot \mathbf{N} = \frac{2}{\sqrt{1+4t^2}} .$$

10. A particle moves in space according to the formula $\mathbf{X}(t) = t\mathbf{I} + e^{2t}\mathbf{J} - 2e^t\mathbf{K}$. Find the normal acceleration at the point $t = 0$.

Answer. Differentiate;

$$\mathbf{V}(t) = \mathbf{I} + 2e^{2t}\mathbf{J} - 2e^t\mathbf{K} , \quad \mathbf{A}(t) = 4e^{2t}\mathbf{J} - 2e^t\mathbf{K} .$$

Now, evaluate at $t = 0$ (don't work with the general formulas!) to find

$$\mathbf{V} = \mathbf{I} + 2\mathbf{J} - 2\mathbf{K}, \ \mathbf{A} = 4\mathbf{J} - 2\mathbf{K} .$$

Then $|\mathbf{V}| = \sqrt{9} = 3$ and since $\mathbf{T} = \mathbf{V}/|\mathbf{V}|$ we have

$$\mathbf{A} \cdot \mathbf{T} = \frac{0+8+4}{3} = 4 ,$$

$$\mathbf{A} - (\mathbf{A} \cdot \mathbf{T})\mathbf{T} = -\frac{4}{3}\mathbf{I} + \frac{4}{3}\mathbf{J} + \frac{2}{3}\mathbf{K} ,$$

so $a_N = |\mathbf{A} - (\mathbf{A} \cdot \mathbf{T})\mathbf{T}| = \frac{\sqrt{16+16+4}}{3} = 2$.