Mathematics 2210-90 Calculus III, Final Examination Jul 29,30,2003

1. Find the equation of the line through the origin and orthogonal to the plane through the points $P(2,-1,0), Q(1,2,1)$ and $R(3,0,2)$.

Solution. By "the equation of the line" we mean either the parametric equation or the symmetric equationsw; whichever is easier for you. The vectors $\overrightarrow{PQ}= -\mathbf{I}+3\mathbf{J}+\mathbf{K}$, $\overrightarrow{QR}= 2\mathbf{I}-2\mathbf{J}+\mathbf{K}$ lie on the plane, so are orhogonal to the line L which we seek. Thus $\overrightarrow{PQ} \times \overrightarrow{QR}= 5\mathbf{I}+3\mathbf{J}-4\mathbf{K}$ is in the direction of the line. Since (0,0,0) is on the line the equation (symmetric or parametric) is

$$\frac{x}{5}=\frac{y}{3}=\frac{z}{-4} \quad \text{or} \quad \mathbf{X}(t)=5t\mathbf{I}+3t\mathbf{J}-4t\mathbf{K} \ .$$

2. A particle moves through space as a function of time: $\mathbf{X}(t)=t\mathbf{I}+\ln t\mathbf{J}+t^2\mathbf{K}$. Find the tangential and normal components of the acceleration when $t=1$.

Solution. First, we differentiate:

$$\mathbf{V}=\mathbf{I}+\frac{1}{t}\mathbf{J}+2t\mathbf{K} \ , \quad \mathbf{A}=-\frac{1}{t^2}\mathbf{J}+2\mathbf{K} \ .$$

Evaluating at $t=1$: $\mathbf{V}=\mathbf{I}+\mathbf{J}+2\mathbf{K}$, $\mathbf{A}=-\mathbf{J}+2\mathbf{K}$, and $ds/dt=\sqrt{6}$. Then

$$a_T=\mathbf{A}\cdot\mathbf{T}=(-\mathbf{J}+2\mathbf{K})\cdot(\frac{\mathbf{I}+\mathbf{J}+2\mathbf{K}}{\sqrt{6}})=\frac{3}{\sqrt{6}} \ .$$

Finally

$$a_N\mathbf{N}=\mathbf{A}-a_T\mathbf{T}=-\mathbf{J}+2\mathbf{K}-\frac{3}{\sqrt{6}}(\frac{\mathbf{I}+\mathbf{J}+2\mathbf{K}}{\sqrt{6}})=\frac{1}{2}(-\mathbf{I}-3\mathbf{J}+2\mathbf{K}) \ ,$$

so $a_N=\sqrt{14}/2$.

3. Find the equation of the tangent plane to the surface $x^2+y^2+3xy+2xz=5$ at the point (1,-1,3).

Solution. Looking at this as a level set $f(x,y,z)=5$, we see that the normal to the tangent plane is

$$\nabla f=(2x+3y+2z)\mathbf{I}+(2y+3x)\mathbf{J}+2x\mathbf{K} \ .$$

Evaluating at (1,-1,3), we get $\mathbf{N}=5\mathbf{I}+\mathbf{J}+2\mathbf{K}$, so the equation of the plane is

$$5(x-1)+(y+1)+2(z-3)=0 \ , \quad \text{or} \quad 5x+y+2z=10 \ .$$

4. Find the maximum value of $3x+2y+z$ on the ellipsoid $x^2+2y^2+3z^2=1$.

Solution. Here we turn to Lagrange multipliers, with the objective function $f(x, y, z) = 3x + 2y + z$, and constraint $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 1$. The gradients are

$$\nabla f = 3\mathbf{I} + 2\mathbf{J} + \mathbf{K}\ , \quad \nabla g = 2x\mathbf{I} + 4y\mathbf{J} + 6z\mathbf{K}\ .$$

The Lagrange equations are (replacing ∇g by $\nabla g/2$).

$$3 = \lambda x\ , \quad 2 = 2\lambda y\ , \quad 1 = 3\lambda z\ , \quad x^2 + 2y^2 + 3z^2 = 1\ .$$

Solve for x, y, z in terms of λ, and put that in the last equation:

$$\frac{9}{\lambda^2} + \frac{2}{\lambda^2} + \frac{3}{9\lambda^2} = 1\ .$$

This leads to $\lambda^2 = 34/3$, so $\lambda = \pm\sqrt{34/3}$. Clearly the maximum is found by taking the positive root, at which point we find

$$x = 3\sqrt{\frac{3}{34}}\ , \quad y = \sqrt{\frac{3}{34}}\ , \quad z = \frac{1}{3}\sqrt{\frac{3}{34}} \quad \text{and} \quad f(x, y, z) = (9 + 2 + \frac{1}{3})\sqrt{\frac{3}{34}} = \sqrt{\frac{34}{3}}$$

5. Find the center of mass of the lamina in the upper half plane bounded by the circle $x^2 + y^2 = 1$ if the density is $\delta(x, y) = x^2 + y^2$.

Solution. Since the region R and the density are symmetric about the line $x = 0$, $\bar{x} = 0$. To find $\bar{y}$, we compute in polar coordinates. The region is given by the equations $0 \le \theta \le \pi,\ 0 \le r \le 1$, and $\delta = r^2$. Thus

$$Mass = \int\int_R \delta dA = \int_0^\pi \int_0^1 r^2 r dr d\theta = \frac{\pi}{4}\ ,$$

$$Mom_{\{y=0\}} = \int\int_R y\delta dA = \int_0^\pi \int_0^1 r^4 \sin\theta dr d\theta = \frac{1}{5}(-\cos\theta)|_0^\pi = \frac{2}{5}\ .$$

Thus $\bar{y} = (2/5)/(\pi/4) = 8/(5\pi)$, and the center of mass is at $(0, 8/(5\pi))$.

6. Given the vector field $\mathbf{F}(x, y) = (y - x^3)\mathbf{I} + (x - y^3)\mathbf{J}$, a) find a function f whose gradient is $\mathbf{F}$. b) Calculate div $\mathbf{F}$.

Solution. Solve $\partial f/\partial x = (y - x^3)$: $f(x, y) = xy - x^4/4 + \phi(y)$ for some unknown function $\phi(y)$. Now set $\partial f/\partial y = x - y^3$:

$$x - \phi'(y) = x - y^3\ , \quad \text{so that} \quad \phi'(y) = -y^3 \quad \text{and} \quad \phi = -\frac{y^4}{4} + C\ .$$

Thus

$$f(x, y) = xy - \frac{x^4}{4} - \frac{y^4}{4} + C\ ,$$

and div $\mathbf{F} = -3(x^2 + y^2)$.

7. Let D be the region in the upper half plane bounded by the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 9$. Let C be the boundary of D traversed counterclockwise. Find

$$\int_C y^2 dx + 3xydy \ .$$

Solution. By Green's theorem, this equals

$$\int \int_D (3y - 2y) dA = \int \int_D y dx dy$$

where D is the region in the upper half plane bounded by the circles of radius 1, 3 respectively. Using polar coordinates, this is

$$\int_0^\pi \int_1^3 r \sin\theta r dr d\theta = -\cos\theta\Big|_0^\pi \cdot \frac{r^3}{3}\Big|_1^3 = \frac{52}{3} \ .$$

8. Let C be the circle $x^2 + y^2 = 16$, oriented counterclockwise. Calculate

$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \ .$$

Solution. We can't use Green's theorem, because the vector field is not defined at the origin, which is a point in the region bounded by C. So we calculate directly, using the parametrization

$$x = 4\cos\theta \ , \quad y = 4\sin\theta \ , \quad dx = -4\sin\theta d\theta \ , \quad dy = 4\cos\theta d\theta \ ,$$

for $0 \le \theta \le 2\pi$. Since $x^2 + y^2 = 16$ on C we have

$$\oint_C \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \frac{1}{16} \int_0^{2\pi} (-4\sin\theta)(-4\sin\theta d\theta) + (4\cos\theta)(4\cos\theta d\theta)$$

$$= \frac{1}{16} \int_0^{2\pi} 16 d\theta = 2\pi \ .$$