Mathematics 2210 Calculus III, Final Examination Answers

Part I. Do FIVE (5) of these first 7 problems.

1. Find the distance of the point $P(2,1,1)$ from the plane given by the equation $x+y-z=1$.

Solution. Choose a point Q on the plane,say $Q(1,1,1)$. Now, the distance of P to the plane is the length of the projection of the vector $\mathbf{V}$ from P to Q in the direction of the normal $\mathbf{N}$ to the plane. Since $\mathbf{V}=\mathbf{I}$, and $\mathbf{N}=\mathbf{I}+\mathbf{J}-\mathbf{K}$, this is

$$\frac{|\mathbf{V}\cdot\mathbf{N}|}{|\mathbf{V}||\mathbf{N}|}=\frac{1}{\sqrt{3}}\ .$$

2. A particle moves in the plane as a function of time: $\mathbf{X}(t)=(2t+1)\mathbf{I}+(t^2-2t+3)\mathbf{J}$. Find the tangential and normal components of the acceleration.

Solution. First we differentiate:

$$\mathbf{V}=\frac{d\mathbf{X}}{dt}=2\mathbf{I}+(2t-2)\mathbf{J}\ ,\quad \mathbf{A}=\frac{d\mathbf{V}}{dt}=2\mathbf{J}\ .$$

Now, we calculate the unit tangent and normal:

$$\mathbf{T}=\frac{\mathbf{V}}{|\mathbf{V}|}=\frac{\mathbf{I}+(t-1)\mathbf{J}}{\sqrt{1+(t-1)^2}}\ ,\quad \mathbf{N}=\mathbf{V}^{\perp}=\frac{(t-1)\mathbf{I}-\mathbf{J}}{\sqrt{1+(t-1)^2}}\ .$$

Finally,

$$a_T=\mathbf{A}\cdot\mathbf{T}=\frac{2(t-1)}{\sqrt{1+(t-1)^2}}\ ,\quad a_N=\mathbf{A}\cdot\mathbf{N}=\frac{2}{\sqrt{1+(t-1)^2}}\ .$$

3. Find the points where the ellipse $x^2+2xy+10y^2=63$ has a horizontal tangent.

Solution. The tangent is horizontal where $dy/dx=0$. We calculate the derivative by taking the differential of the defining equation:

$$2xdx+2ydx+2xdy+20ydy=0\quad\text{or}\quad(2x+2y)dx+(2x+20y)dy=0\ .$$

Now, $dy/dx=0$ at those points where the coefficient of dx is zero, that is, where $y=-x$. Substituting that in the defining equation gives us: $x^2-2x^2+10x^2=63$, or $9x^2=63$, which has the solution $x=\pm\sqrt{7}$. Thus the points sought are $(\sqrt{7},\sqrt{7}),(-\sqrt{7},\sqrt{7})$.

If instead we want to work with gradients, we calculate $\nabla f=-(2x+2y)\mathbf{I}+(2x+20y)\mathbf{J}$. Since the gradient is orthogonal to the level set, we want ∇f to be vertical, that is, collinear with $\mathbf{J}$, which brings us again to the equation $2x+2y=0$.

4. Let $f(x,y) = x^2 + xy + y^2$. A particle is moving through the plane so that its position at time t is $\mathbf{X}(t) = \sin t\mathbf{I} + \cos t\mathbf{J}$. Find df/dt when $t = \pi/3$.

Solution. First we calculate the derivatives:

$$\nabla f = (2x+y)\mathbf{I} + (x+2y)\mathbf{J}\ , \quad \frac{d\mathbf{X}}{dt} = \cos t\mathbf{I} - \sin t\mathbf{J}\ .$$

When $t = \pi/3,\ x = \cos(\pi/3) = 1/2,\ y = \sin(\pi/3) = \sqrt{3}/2$, so

$$\nabla f = (\sqrt{3} + \frac{1}{2})\mathbf{I} + (\frac{\sqrt{3}}{2} + 1)\mathbf{J}\ , \quad \frac{d\mathbf{X}}{dt} = \frac{1}{2}\mathbf{I} - \frac{\sqrt{3}}{2}\mathbf{J}\ ,$$

and

$$\frac{df}{dt} = \nabla f \cdot \frac{d\mathbf{X}}{dt} = ((\sqrt{3} + \frac{1}{2})\mathbf{I} + (\frac{\sqrt{3}}{2} + 1)\mathbf{J}) \cdot (\frac{1}{2}\mathbf{I} - \frac{\sqrt{3}}{2}\mathbf{J}) = -\frac{1}{2}\ .$$

One can first make the substitution to calculate:

$$f(\mathbf{X}(t)) = \sin^2 t + \sin t\cos t + \cos^2 t = 1 + \sin t\cos t = 1 + \frac{1}{2}\sin(2t)\ .$$

Now, differentiate:

$$\frac{df}{dt} = \frac{1}{2}\sin(2t)(2) = \cos(2t)\ .$$

Finally, at $t = \pi/3,\ f'(t) = -1/2$.

5. Find a vector perpendicular to the line given by the symmetric equations

$$\frac{x-1}{2} = \frac{y+1}{4} = \frac{z}{5}$$

lying in the plane given by $2x + y + z = 0$.

Solution. Let $\mathbf{W}$ be the vector we are looking for. The vector $\mathbf{L} = 2\mathbf{I} + 4\mathbf{J} + 5\mathbf{K}$ lies in the direction of the line, so is perpendicular to $\mathbf{W}$. The normal to the plane $\mathbf{N} = 2\mathbf{I} + \mathbf{J} + \mathbf{K}$ is also perpendicular to $\mathbf{W}$. So, we can take $\mathbf{W} = \mathbf{L} \times \mathbf{N}$:

$$\mathbf{W} = \mathbf{L} \times \mathbf{N} = \det\begin{pmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix} = (4-5)\mathbf{I} + (10-2)\mathbf{J} + (2-8)\mathbf{K} = -\mathbf{I} + 8\mathbf{J} - 6\mathbf{K}\ .$$

6. Find the maximum value of $3x^2 - y^2 + z$ on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

Solution. We use the method of Lagrange multipliers. First, we take the gradients of the two functions

$$\nabla f = 6x\mathbf{I} - 2y\mathbf{J} + \mathbf{K}\ , \quad \nabla g = 2x\mathbf{I} + 4y\mathbf{J} + 6z\mathbf{K}\ .$$

The Lagrange equations are

$$6x = 2\lambda x \ , \quad -2y = 4\lambda y \ , \quad 1 = 6\lambda z \ , \quad x^2 + 2y^2 + 3z^2 = 1 \ .$$

From the third equation, we see that λ and z cannot be zero. Then from the first and second equations we get that $x = 0$ or $y = 0$.

case $x = 0, y \neq 0$: from the second equation, $\lambda = -1/2$ and then from the third equation $z = -1/3$, and now we use the fourth to find y:

$$2y^2 + 3\frac{1}{9} = 1 \ , \quad \text{so} \quad y = \pm\frac{1}{\sqrt{3}} \ , \quad \text{and} \quad f(0, \pm\frac{1}{\sqrt{3}}, -\frac{1}{3}) = 0 \ .$$

case $x \neq 0, y = 0$: from the first equation, $\lambda = 3$, and from the third, $z = 1/18$. Now, the computation with the last equation gives us $x = \sqrt{107}/18$, and the value of f at this point is $110/36$.

case $x = 0, y = 0$: The last equation then gives us $z = \pm 1/\sqrt{3}$, and the value of f is $\pm 1/\sqrt{3}$. The maximum value is the largest of these calculated values: $110/36$.

7. Let R be the part of the unit sphere in the first octant. Suppose that it is filled with a material whose density at the point (x, y, z) is given by $\delta(x, y, z) = xyz$. Find the total mass.

Solution. We use spherical coordinates. The region is that bounded by $\rho \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2$, $dV = \delta\rho^2 \sin\phi d\rho d\phi d\theta$, and

$$\delta(\rho, \phi, \theta) = (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi) = \rho^3\cos\theta\sin\theta\sin^2\phi \ cos\phi \ .$$

Thus

$$Mass = \int\int\int \delta dV = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \rho^5 \cos\theta\sin\theta\sin^3\phi\cos\phi d\rho d\theta d\phi$$

$$= \int_0^1 \rho^5 d\rho \int_0^{\pi/2} \cos\theta\sin\theta d\theta \int_0^{\pi/2} \sin^3\phi\cos\phi d\phi = \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{48} \ .$$

Part II. Do ALL three problems.

8. Given the vector field $\mathbf{F}(x, y) = (x^2 - y)\mathbf{I} + (y^2 - x)\mathbf{J}$, a) find a function f whose gradient is $\mathbf{F}$.

Solution.

$$\text{From} \quad \frac{\partial f}{\partial x} = x^2 - y \quad \text{we get} \quad f = \frac{x^3}{3} - xy + \phi$$

where ϕ is some unknown function of y. Setting

$$\frac{\partial f}{\partial y} = y^2 - x \quad \text{we get} \quad -x + \phi' = y^2 - x \ ,$$

so $\phi = y^3/3$. Thus

$$f(x,y) = \frac{x^3}{3} - xy + \frac{y^3}{3} + C \ .$$

b) We calculate div $\mathbf{F} = 2x + 2y$.

9. Let D be the region in the plane bounded by the circle $x^2 + y^2 = 9$. Let C be the boundary of D traversed counterclockwise. Find

$$\int_C y^2 dx + 2xydy \ .$$

Solution. Use Green's theorem, where C is noted to be the boundary of the disc D of radius 3:

$$\int_C y^2 dx + 2xydy = \int\int_D (2y - 2y)dxdy = 0 \ .$$

Alternatively, parametrize the circle by $\mathbf{X}(t) = 3\cos t\mathbf{I} + 3\sin t\mathbf{J}, 0 \le t \le 2\pi$. Then

$$\int_C y^2 dx + 2xydy = \int_0^{2\pi} 9\cos^2 t(3\cos tdt) + 9\sin^2 t(-3\sin tdt) = 27\int_0^{2\pi} (\cos^3 t - \sin^3 t)dt$$

$$= 27\int_0^{2\pi} [(1-\sin^2 t)\cos t - (1-\cos^2 t)\sin t]dt = 27(\sin t - \frac{\sin^3 t}{3} + (\cos t + \frac{\cos^3 t}{3})\Big|_0^{2\pi} = 0 \ .$$

10. Let C be the curve given parametrically by

$$\mathbf{X}(t) = t\mathbf{I} + t^2\mathbf{J} + t^3\mathbf{K}$$

for t running from 0 to 2. For the vector field $\mathbf{F} = x\mathbf{I} + y\mathbf{J} + 2x\mathbf{K}$, find $\int_C \mathbf{F}\cdot\mathbf{T}ds$.

Solution. First we note that $d\mathbf{X} = \mathbf{T}ds$, and $d\mathbf{X} = (\mathbf{I} + 2t\mathbf{J} + 3t^2\mathbf{K})dt$. Along the curve, in terms of t, $\mathbf{F} = t\mathbf{I} + t^2\mathbf{J} + 2t\mathbf{K}$. Thus

$$\int_C \mathbf{F}\cdot\mathbf{T}ds = \int_0^2 (t\mathbf{I} + t^2\mathbf{J} + 2t\mathbf{K})\cdot(\mathbf{I} + 2t\mathbf{J} + 3t^2\mathbf{K})dt$$

$$= \int_0^2 (t + 2t^3 + 6t^3)dt = (\frac{t^2}{2} + 2t^4)_0^2 = 34$$