Calculus III, Mathematics 2210-90

Examination 2, July 17,19, 2003: Answers

1. Find
$$\int\int_R xy^2 dxdy$$
where R is the region in the plane given by the inequalities $x \geq 0,\ x^2 - 1 \leq y \leq 1 - x^2$.

Solution. This is a type 1 domain given by the inequalities $0 \leq x \leq 1,\ x^2 - 1 \leq y \leq 1 - x^2$. (You must draw the diagram to see that we run out of domain at $x = 1$. Thus the integral is calculated as the iterated integral

$$\int_0^1 \int_{x^2-1}^{1-x^2} xy^2 dydx \ .$$

The inner integral is

$$\int_{x^2-1}^{1-x^2} xy^2 dy = \frac{xy^3}{3}\Big|_{x^2-1}^{1-x^2} = \frac{2}{3}(1-x^2)^3 \ .$$

Then, the answer is

$$\int_0^1 \frac{2}{3}x(1-x^2)^3 dx = -\frac{1}{3}\int_1^0 u^3 du = \frac{1}{12} \ ,$$

using the substituion $u = 1 - x^2,\ du = -2xdx$.

Noting the symmetry in the x-axis, could also view upper part as the domain as type 2: $0 \leq y \leq 1,\ 0 \leq x \leq \sqrt{1-y}$. Then the inner integral is

$$\int_0^{\sqrt{1-y}} y^2 x dx = \frac{1}{2}y^2 x^2\Big|_0^{\sqrt{1-y}} = \frac{1}{2}(1-y)y^2 \ .$$

The answer we want is twice this, integrated in y from 0 to 1:

$$\int_0^1 (y^2 - y^3)dy = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \ .$$

2. Find the volume under the plane $z = x + 2y + 1$ over the triangle bounded by the lines $y = 0,\ x = 1,\ y = 2x$

Solution. If we sweep out along the x-axis, we can calculate the volume as $\int_0^1 A(x)dx$, where, for fixed x, $A(x)$ is the area under the curve $z = x + 2y + 1$ over the line segment at x in the triangle. This is the line from $y = 0$ to $y = 2x$. Thus

$$A(x) = \int_0^{2x} (x + 2y + 1)dy = (xy + y^2 + y)\Big|_0^{2x} = 2x^2 + 4x^2 + 2x = 6x^2 + 2x \ .$$

Then the volume is

$$Volume \ = \int_0^1 [\int_0^{2x} (x+2y+1)dy]dx = \int_0^1 (6x^2+2x)dx = (2x^3+x^2)\big|_0^1 = 3 \ .$$

Now we'll do the calculation by sweeping out along the y axis first. Now, y ranges from 0 to 2, and for fixed y, x ranges from $y/2$ to 1. This computation leads to

$$Volume \ = \int_0^2 [\int_{y/2}^1 (x+2y+1)dx]dy \ .$$

The inner integral is

$$\int_{y/2}^1 (x+2y+1)dy = (\frac{x^2}{2}+2xy+x)\big|_{y/2}^1 = \frac{3}{2}+\frac{3}{2}y-\frac{9}{8}y^2 \ .$$

We then get

$$Volume \ = \int_0^2 (\frac{3}{2}+\frac{3}{2}y-\frac{9}{8}y^2)dy = (\frac{3}{2}y+\frac{3}{4}y^2-\frac{3}{8}y^3\big|_0^2 = 3+\frac{3}{4}-\frac{3}{4} = 3 \ .$$

3. A lamina (that is, a thin plate) lies over the region in the first quadrant bounded by the circle $x^2+y^2=1$. The density of the material in the lamina is $\delta(x,y)=\sqrt{x^2+y^2}$. What are the coordinates of the center of mass of the lamina?

In polar coordinates, this is the region $R: \ 0 \le \theta \le \pi/2, r \le 1$, and $\delta = r$. Thus

$$Mass = \int\int_R \delta r dr d\theta = \int_0^{\pi/2} \int_0^1 r^2 dr d\theta = \frac{\pi}{6} \ .$$

Now, since $x = r\cos\theta$,

$$Mom_{x=0} = \int\int_R x\delta r dr d\theta = \int_0^{\pi/2} \int_0^1 r^3 \cos\theta dr d\theta = \frac{1}{4}(-\sin\theta)\big|_0^{\pi/2} = \frac{1}{4} \ .$$

Thus $\bar{x} = (1/4)/(\pi/6) = 3/(2\pi)$. Now, by the symmetry, $\bar{y} = \bar{x}$.

You may have thought to define a new concept: the moment in the polar coordinate r y replacing x by r in the formula. That is off by the factor $\pi/2$, for the contribution of the infinitesimal arc at a value of r is the length of the arc, which is $\pi r/2$, not r.

4. Find the volume of the elliptical cone bounded by the surfaces $0 \le z \le 5$, $z^2 = x^2+4y^2$,.

Solution. If you don't change coordinates, this leads to an integral you don't really want to see. But the change of coordinates $u=x$, $v=2y$ changes the function to $z=\sqrt{u^2+v^2}$. In these coordinates the solid we are interested in is the region lying over the disc $0 \le \sqrt{u^2+v^2} \le 5$, and between the surfaces $z = 5$ and $z = \sqrt{u^2+v^2}$. By writing the change of variables as $x = u,\ y = v/2$, we easily compute that the Jacobian is 1/2. Thus

$$Volume = \int\int_{\sqrt{u^2+v^2}\le 5} (5 - \sqrt{u^2+v^2})(\frac{1}{2})dudv = \frac{1}{2}\int_0^{2\pi}\int_0^5 (5-r)rdrd\theta = \frac{125}{6}\pi \ ,$$

by changing to polar coordinates in uv-space.

5. The upper hemisphere of radius 3, $H:\ x^2+y^2+z^2 \le 9,\ z \ge 0$ is filled with a material whose density at the point (x, y, z) is $\delta(x, y, z) = (x^2+y^2+z^2)^{3/2}$. Find the mass of this solid.

Solution. Turn to spherical coordinates. The H is the region $0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi, \rho \le 3$, and $\delta = \rho^3$. Thus

$$Mass = \int\int\int_H \delta dV = \int_)^{\pi/2}\int_0^{2\pi}\int_0^3 \rho^3\rho^2 \sin\phi d\rho d\theta d\phi = (1)(2\pi)(\frac{3^6}{6}) = 3^5\pi \ .$$