Calculus III, Mathematics 2210-90

Examination 3, Nov 13,15, 2003, Answers

You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. Find $$\int\int_R (1-x^2-y^2)dxdy$$

where R is the region in the plane bounded by the curves $y=0,\ y=x^2,\ x=1$.

Solution. This is a type 1 region: $0 \le x \le 1,\ 0 \le y \le x^2$. Thus

$$\int\int_R (1-x^2-y^2)dxdy = \int_0^1 [\int_0^{x^2} (1-x^2-y^2)dy]dx \ .$$

The inner integral is

$$\int_0^{x^2} (1-x^2-y^2)dy = [y - x^2y - \frac{y^3}{3}]_0^{x^2} = x^2 - x^4 - \frac{x^6}{3} \ .$$

Then our answer is

$$\int_0^1 (x^2 - x^4 - \frac{x^6}{3})dx = \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{35-21-5}{21} = \frac{3}{7} \ .$$

2. Find the volume of the solid under the surface $z = 9 - x^2 - y^2$ and over the disk $x^2+y^2 \le 9$, and between the planes $y=0$ and $y=x$.

Solution. It looks like a good idea to switch to cylindrical coordinates: we want the volume of the solid under the surface $z = 9 - r^2$ lying over the segment $R:\ r \le 3,\ 0 \le \theta \le \pi/4$. This is

$$\int\int_R (9-r^2)dA = \int_0^{\pi/4}\int_0^3 (9-r^2)rdrd\theta = \frac{\pi}{4}\int_0^3 (9r-r^3)dr$$

$$= \frac{\pi}{4}(\frac{9}{2}r^2 - \frac{r^4}{4})_0^3 = \frac{\pi}{4}(\frac{81}{2} - \frac{81}{4}) = \frac{81\pi}{16}$$

3. Find the area of the piece of the surface $z = x^2 - y^2$ lying over the disk $D:\ x^2+y^2 \le 4$.

Solution. We know that $dS = \sqrt{1+z_x^2+z_y^2}dA$. Since $z_x = 2z,\ z_y = -2y$, this gives $dS = \sqrt{1+4x^2+4y^2}dA = \sqrt{1+4r^2}rdrd\theta$ in polar coordinates. Thus the area is

$$\int\int_D \sqrt{1+4r^2}rdrd\theta = \int_0^{2\pi}[\int_0^2 \sqrt{1+4r^2}rdr]d\theta = 2\pi\int_0^2 \sqrt{1+4r^2}rdr \ .$$

Making the substitution $u = 1 + 4r^2,\ du = 8rdr$, we have

$$\int_0^2 \sqrt{1+4r^2}rdr = \frac{1}{8}\int_1^{17} u^{1/2}du = \frac{1}{8}(\frac{2}{3}(17^{3/2}-1)) = \frac{1}{12}(17^{3/2}-1)\ .$$

4. Find the area of the parallelogram Π bounded by the lines

$$2x + y = 1,\ 2x + y = 3,\ y = x,\ y = x + 4\ .$$

Solution. If we let $u = 2x + y,\ v = y - x$ this is the region R given by the inequalities $1 \le u \le 3,\ 0 \le v \le 4$. Thus the area is

$$\int\int_\Pi dxdy = \int\int_R \frac{\partial(x,y)}{\partial(u,v)}dudv\ .$$

Now, we can solve for $x,\ y$ in terms of $u,\ v$:

$$x = \frac{u}{3} - \frac{v}{3}\ , \quad y = \frac{u}{3} + \frac{2v}{3}\ .$$

We calculate $x_u = 1/3,\ x_v = -1/3,\ y_u = 1/3,\ y_v = 2/3$, so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}\frac{2}{3} - \frac{1}{3}(-\frac{1}{3}) = \frac{1}{3}\ .$$

Thus the answer is

$$\int_1^3\int_0^4 \frac{1}{3}dudv = \frac{1}{3}(3-1)(4-0) = \frac{8}{3}\ .$$

5. The region R in 3 dimensions bounded by the planes

$$x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = x + y$$

is filled with an inhomogeneous mud whose density is $\delta(x,y,z) = 2 - z$. Find the mass of mud in this region.

Solution. R can be set up for integration as defined by the inequalities $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le x + y$. Then the mass is

$$\int\int\int_R \delta dV =]int_0^1 \int_0^1 \int_0^{x+y} (2-z)dzdydx\ .$$

We now compute the integrals successively, starting with the innermost:

$$\int_0^{x+y}(2-z)dz = 2z - \frac{z^2}{2}\Big|_0^{x+y} = 2x + 2y - \frac{1}{2}(x^2 + 2xy + y^2)\ ,$$

$$\int_0^1 (2x + 2y - xy - \frac{x^2}{2} - \frac{y^2}{2})dy = 2xy + y^2 - x\frac{y^2}{2} - \frac{x^2}{2}y - \frac{y^3}{6}\Big|_0^1$$

$$= 2x + 1 - x - \frac{x^2}{2} - \frac{1}{6} = -\frac{x^2}{2} + x + \frac{5}{6}\ ,$$

$$\int_0^1 (-\frac{x^2}{2} + x + \frac{5}{6})dx = -\frac{x^3}{6} + -\frac{x^2}{2} + \frac{5}{6}x\Big|_0^1 = -\frac{1}{6} + \frac{1}{2} + \frac{5}{6} = \frac{7}{6}\ .$$