Calculus III
Exam 2, Answers

You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. A conic in the plane is given by the equation

$$2x^2 - 2xy + y^2 + 2x - y = 100 \ .$$

a) What conic is it?

Answer. $B^2 - 4AC = (-2)^2 - 4(2)(1) = -4 < 0$, so it is an ellipse.

b) At what angle to the x-axis are the axes of the conic?

Answer. If θ is the angle of one of the axes with the x-axis, we know that

$$\tan(2\theta) = \frac{B}{A-C} = \frac{-2}{2-1} = -2 \ ,$$

so $\theta = (1/2)\arctan(-2) = -.3524\pi$ radians. The other axis is perpendicular to this axis, so is at an angle $\theta + \pi/2 = .1476\pi$ radians.

---

2. Let $f(x,y) = x^2 - 3xy - y^2$.

Answer. a) $\nabla f = (2x - 3y)\mathbf{I} - (3x + 2y)\mathbf{J}$.

b) At (1,2), $\nabla f = -4\mathbf{I} - 7\mathbf{J}$. The direction of maximum increase is that of ∇f, so is

$$\mathbf{U} = -\frac{4\mathbf{I} + 7\mathbf{J}}{\sqrt{65}} \ .$$

c) Find the equation of the plane tangent to the surface $z = f(x,y)$ at the point (1,2,-9).

The equation of the tangent plane is $z - (-9) = \nabla f \cdot ((x-1)\mathbf{I} + (y-2)\mathbf{J})$, or

$$z + 9 = -4(x-1) - 7(y-2)$$

which simplifies to $4x + 7y + z = 9$.

---

3. Let $f(x,y,z) = x\sin(yz)$. Let the curve γ be given by the equation $\mathbf{X}(t) = 2t\mathbf{I} - t\mathbf{J} + \mathbf{K}$. Let $g(t) = f(\mathbf{X}(t))$. Do the following calculations at the value $t = \pi/2$;

Answer. a) $\nabla f = \sin(yz)\mathbf{I} + xz\cos(yz)\mathbf{J} + xy\cos(yz)\mathbf{K}$,

which evaluates at $t = \pi/2$ to

$$\nabla f = \sin(-\frac{\pi}{2})\mathbf{I} + \pi\cos(-\frac{\pi}{2})\mathbf{J} - \frac{\pi^2}{2}\cos(-\frac{\pi}{2})\mathbf{K} = -\mathbf{I} \ .$$

b) $\dfrac{d\mathbf{X}}{dt} = 2\mathbf{I} - \mathbf{J}$,

c) $\frac{dg}{dt} = \nabla f \cdot \frac{d\mathbf{X}}{dt} = -2$.

---

4. Let $f(x,y) = 3x^2y + 3xy$.

Answer. a) $\nabla f = (6xy + 3y)\mathbf{I} + (3x^2 + 3x)\mathbf{J}$.

b) What are the critical points of f?

Answer. These are the solutions of the equations $f_x = 0$, $f_y = 0$, or

$$6xy + 3y = 0 \ , \quad 3x^2 + 3x = 0 \ .$$

From the second equation, either $x = 0$ or $x = -1$. From the first equation, for both values of x we get $y = 0$. Thus the critical points are (0,0) and (-1,0).

c) What kind of critical points are they?

Answer. We have $f_{xx} = 6y$, $f_{xy} = 6x + 3$, $f_{yy} = 0$, so $D = -(6x + 3)^2 < 0$ at both points. Thus both points are saddle points.

---

5. Find the maximum value of xy on the curve $x^2 + 2y^2 = 1$.

Answer. Let $f(x,y) = xy$ and $g(x,y) = x^2 + 2y^2$. Using Lagrange multipliers, the maximum is taken at some point where $\nabla f = \lambda \nabla g, g(x,y) = 1$. We differentiate to obtain

$$\nabla f = y\mathbf{I} + x\mathbf{J} \ , \quad \nabla g = 2x\mathbf{I} + 4y\mathbf{J}$$

so the Lagrange equations are

$$y = 2\lambda x \ , \quad x = 4\lambda y \ , \quad x^2 + 2y^2 = 1 \ .$$

Substituting the first in the second we obtain $x = 8\lambda^2 x$, so either $x = 0$ or $\lambda = \pm 8^{-1/2}$.

Case $x = 0$. From the last equation we find $y = \pm 2^{-1/2}$, and $f(x,y) = 0$.

Case $\lambda = \pm 8^{-1/2}$. Then $y = 2(8^{-1/2})x$; putting that in the last equation gives

$$x^2 + 2\frac{4}{8}x^2 = 1 \ , \quad \text{or} \quad 2x^2 = 1$$

giving the solutions $x = \pm 2^{-1/2}$, $y = \pm 1/2$. The values of f at these points are

$$\pm \frac{1}{2\sqrt{2}} \quad \text{so the maximum is} \quad \frac{1}{2\sqrt{2}} \ ,$$

taken at the points $\pm(1/\sqrt{2}, 1/2)$. Another way to go is to eliminate 2λ from the first two Lagrange equations; this leads to

$$\frac{y}{x} = \frac{x}{2y} \ , \quad \text{or} \quad x^2 = 2y^2 \ .$$

Substituting this in the last equation gives $4y^2 = 1$, which leads easily to the solution.