You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. A conic in the plane is given by the equation

\[ 2x^2 - 2xy + y^2 + 2x - y = 100. \]

a) What conic is it?

**Answer.** \( B^2 - 4AC = (-2)^2 - 4(2)(1) = -4 < 0 \), so it is an ellipse.

b) At what angle to the \( x \)-axis are the axes of the conic?

**Answer.** If \( \theta \) is the angle of one of the axes with the \( x \)-axis, we know that

\[ \tan(2\theta) = \frac{B}{A-C} = \frac{-2}{2-1} = -2, \]

so \( \theta = (1/2)\arctan(-2) = -0.3524\pi \) radians. The other axis is perpendicular to this axis, so is at an angle \( \theta + \pi/2 = 1.1476\pi \) radians.

2. Let \( f(x,y) = x^2 - 3xy - y^2. \)

**Answer.**

a) \( \nabla f = (2x - 3y)\mathbf{I} - (3x + 2y)\mathbf{J}. \)

b) At \((1,2)\), \( \nabla f = -4\mathbf{I} - 7\mathbf{J}. \) The direction of maximum increase is that of \( \nabla f \), so is

\[ \mathbf{U} = \frac{-4\mathbf{I} + 7\mathbf{J}}{\sqrt{65}}. \]

c) Find the equation of the plane tangent to the surface \( z = f(x,y) \) at the point \((1,2,-9)\).

The equation of the tangent plane is \( z - (-9) = \nabla f \cdot ((x - 1)\mathbf{I} + (y - 2)\mathbf{J}), \) or

\[ z + 9 = -4(x - 1) - 7(y - 2) \]

which simplifies to \( 4x + 7y + z = 9. \)

3. Let \( f(x,y,z) = x \sin(yz) \). Let the curve \( \gamma \) be given by the equation \( \mathbf{X}(t) = 2t\mathbf{I} - t\mathbf{J} + \mathbf{K}. \) Let \( g(t) = f(\mathbf{X}(t)) \).

Do the following calculations at the value \( t = \pi/2; \)

**Answer.**

a) \( \nabla f = \sin(yz)\mathbf{I} + xz \cos(yz)\mathbf{J} + xy \cos(yz)\mathbf{K} \),

which evaluates at \( t = \pi/2 \) to

\[ \nabla f = \sin(-\frac{\pi}{2})\mathbf{I} + \pi \cos(-\frac{\pi}{2})\mathbf{J} - \frac{\pi^2}{2} \cos(-\frac{\pi}{2})\mathbf{K} = -\mathbf{I}. \]

b) \[ \frac{d\mathbf{X}}{dt} = 2\mathbf{I} - \mathbf{J}, \]
c) \( \frac{dg}{dt} = \nabla f \cdot \frac{dX}{dt} = -2 \).

4. Let \( f(x,y) = 3x^2y + 3xy \).

**Answer.** a) \( \nabla f = (6xy + 3y)\mathbf{I} + (3x^2 + 3x)\mathbf{J} \).

b) What are the critical points of \( f \)?

**Answer.** These are the solutions of the equations \( f_x = 0, f_y = 0 \), or
\[
6xy + 3y = 0, \quad 3x^2 + 3x = 0.
\]
From the second equation, either \( x = 0 \) or \( x = -1 \). From the first equation, for both values of \( x \) we get \( y = 0 \). Thus the critical points are \((0,0)\) and \((-1,0)\).

c) What kind of critical points are they?

**Answer.** We have \( f_{xx} = 6y, f_{xy} = 6x + 3, f_{yy} = 0 \), so \( D = -(6x + 3)^2 < 0 \) at both points. Thus both points are saddle points.

5. Find the maximum value of \( xy \) on the curve \( x^2 + 2y^2 = 1 \).

**Answer.** Let \( f(x,y) = xy \) and \( g(x,y) = x^2 + 2y^2 \). Using Lagrange multipliers, the maximum is taken at some point where \( \nabla f = \lambda \nabla g, g(x,y) = 1 \). We differentiate to obtain
\[
\nabla f = y\mathbf{I} + x\mathbf{J}, \quad \nabla g = 2x\mathbf{I} + 4y\mathbf{J}
\]
so the Lagrange equations are
\[
y = 2\lambda x, \quad x = 4\lambda y, \quad x^2 + 2y^2 = 1.
\]
Substituting the first in the second we obtain \( x = 8\lambda^2 x \), so either \( x = 0 \) or \( \lambda = \pm 8^{-1/2} \).
Case \( x = 0 \). From the last equation we find \( y = \pm 2^{-1/2} \), and \( f(x,y) = 0 \).
Case \( \lambda = \pm 8^{-1/2} \). Then \( y = 2(8^{-1/2})x \); putting that in the last equation gives
\[
x^2 + 2^4 \frac{4}{8} x^2 = 1, \quad \text{or} \quad 2x^2 = 1
\]
giving the solutions \( x = \pm 2^{-1/2}, y = \pm 1/2 \). The values of \( f \) at these points are
\[
\pm \frac{1}{2\sqrt{2}}, \quad \text{so the maximum is} \quad \frac{1}{2\sqrt{2}},
\]
taken at the points \( \pm(1/\sqrt{2}, 1/2) \). Another way to go is to eliminate \( 2\lambda \) from the first two Lagrange equations; this leads to
\[
\frac{y}{x} = \frac{x}{2y}, \quad \text{or} \quad x^2 = 2y^2.
\]
Substituting this in the last equation gives \( 4y^2 = 1 \), which leads easily to the solution.