Calculus III, Mathematics 2210-90

Examination 2, Mar 11,13, 2004: Answers

You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. Consider the line L in the plane given by the equation $2x + 5y + 10 = 0$.
a) Find a base $\{\mathbf{U}, \mathbf{V}\}$ with $\mathbf{U}$ parallel to L, and $\mathbf{V}$ counterclockwise to $\mathbf{U}$. By "base" we mean that $\mathbf{U}$ and$\mathbf{V}$ are orthogonal unit vectors.
b) Find the equation of the line in coordinates $\{u, v\}$ relative to the base $\{\mathbf{U}, \mathbf{V}\}$.

Solution. The vector $2\mathbf{I} + 5\mathbf{J}$ is orthogonal to the line, so $5\mathbf{I} - 2\mathbf{J}$ is in the direction of the line. Thus we can take

$$\mathbf{U} = \frac{5\mathbf{I} - 2\mathbf{J}}{\sqrt{29}} \ , \quad \mathbf{V} = \frac{2\mathbf{I} + 5\mathbf{J}}{\sqrt{29}} \ .$$

If $\mathbf{X} = x\mathbf{I} + y\mathbf{J} = u\mathbf{U} + v\mathbf{V}$, then

$$u\mathbf{U} + v\mathbf{V} = u(\frac{5\mathbf{I} - 2\mathbf{J}}{\sqrt{29}}) + v(\frac{2\mathbf{I} + 5\mathbf{J}}{\sqrt{29}}) = (\frac{5u + 2v}{\sqrt{29}})\mathbf{I} + (\frac{-2u + 5v}{\sqrt{29}})\mathbf{J} = x\mathbf{I} + y\mathbf{J} \ ,$$

giving us x and y in terms of u, v. Putting these expressions in the equation gives the u, v equation of the line as

$$2(\frac{5u + 2v}{\sqrt{29}}) + 5(\frac{-2u + 5v}{\sqrt{29}}) + 10 = 0 \ , \quad \text{or} \quad v = -\frac{10}{\sqrt{29}} \ .$$

2. Write down the equations of the paraboloid of revolution $z = x^2 + y^2$ in cylindrical and spherical coordinates.

Solution. In cylindrical coordinates, $x^2 + y^2 = r^2$, so the equation is simply $z = r^2$. In spherical coordinates, this becomes $\rho \cos\phi = (\rho \sin\phi)^2$, or $\rho = \cot\phi \csc\phi$, for $0 < \phi \le \pi/2$.

3. Find the unit vector $\mathbf{U}$ in the direction of maximal change for the function $w = x^3y^2z + xyz^2$ at the point (2,-1,2). What is $D_{\mathbf{U}}w$ at this point?

Solution. The direction of maximal change is that of the gradient. We calculate the gradient:

$$\nabla w = (3x^2y^2z + yz^2)\mathbf{I} + (2x^3yz + xz^2)\mathbf{J} + (x^3y^2 + 2xyz)\mathbf{K} \ .$$

Evaluating at the given point, we have $\nabla w = 20\mathbf{I} - 24\mathbf{J}$. $\mathbf{U}$ is the unit vector in this direction, so $\mathbf{U} = (5\mathbf{I} - 6\mathbf{J})/\sqrt{61}$. At this point $D_{\mathbf{U}}w = |\nabla w| = 4\sqrt{61}$.

4. Find the equation of the tangent plane to the surface

$$x^{1/2} + y^{1/2} - z^{1/2} = 0$$

at the point (4,9,25).

Solution. Take the differential of the defining equation:

$$\frac{1}{2}x^{-1/2}dx + \frac{1}{2}y^{-1/2}dy - \frac{1}{2}z^{-1/2}dz = 0 \ .$$

Multiply by 2 and evaluate at (4,9,25):

$$\frac{dx}{2} + \frac{dy}{3} - \frac{dz}{5} = 0 \ ,$$

and replace the differentials by the increments:

$$\frac{x-4}{2} + \frac{y-9}{3} - \frac{z-25}{5} = 0 \ .$$

When simplified, this comes down to $15x + 10y - 6z = 0$.

5. Let $f(x, y) = x^2y^2 + x + y$.
a) Find the critical points of f.
b) What kind of critical points are they? (maximum, minimum, saddle?)

Solution. $\nabla f = (2xy^2 + 1)\mathbf{I} + (2x^2y + 1)\mathbf{J}$. The equations for $\nabla f = 0$ are

$$2xy^2 + 1 = 0 \ , \quad 2x^2y + 1 = 0 \ .$$

From the first equation either $x = -1/2y^2$. Substituting that in the second equation gives us

$$\frac{2}{4y^4}y + 1 = 0 \quad \text{so that} \quad y = -\frac{1}{2}^{1/3} \quad \text{and} \quad x = -\frac{1}{2}^{1/3} \ .$$

b) We calculate the discriminant: $f_{xx} = 2y^2$, $f_{yy} = 2x^2$ and $f_{xy} = 4xy$. Thus

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4x^2y^2 - 16x^2y^2 < 0$$

so the point is a saddle point.