Calculus III, Mathematics 2210-90

Examination 2, October 16,18, 2003

You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1a. Write down an equation of a hyperbola whose major axis is the line $y = x$.

Solution. $xy = 1$ will do. So will any form of the type $Ax^2 + Bxy + Ay^2 = D$, with $B^2 > 4A^2$. Another idea would be to start with $u^2 - v^2 = 1$, and make the substitution $u = x + y,\ v = x - y$, which leads to $4xy = 1$.

1b. Write down an equation of an ellipse (not a circle) whose major axis is the line $y = x$.

Solution. Any form of the type $Ax^2 + Bxy + Ay^2 = D$, with $B^2 < 4A^2$ will do. Another idea would be to start with $4u^2 + v^2 = 1$, and make the substitution $u = x + y,\ v = x - y$, which leads to $5x^2 + 6xy + 5y^2 = 1$.

2. Let $f(x, y) = x^2 + 3xy + 4y^2$.

a) What is ∇f ?

$$\nabla f = (2x + 3y)\mathbf{I} + (3x + 8y)\mathbf{J}\ .$$

b) Find the equation of the line tangent to the curve $f(x, y) = 14$ at the point (2,1).

Solution. At $\mathbf{X_0} : (2, 1),\ \nabla f = 7\mathbf{I} + 14\mathbf{J}$. The equation of the line is $\nabla f \cdot (\mathbf{X} - \mathbf{X_0}) = 0$, which is $7(x - 2) + 14(y - 1) = 0$, which simplifies to $x + 2y = 4$.

3. Let $w = x^2 - yz$. Let $\mathbf{X}(t) = e^t\mathbf{I} + e^{t+1}\mathbf{J} + e^{2t}\mathbf{K}$ describe the curve Γ. Find the formula for dw/dt along Γ as a function of t.

Solution. $\nabla w = 2x\mathbf{I} - z\mathbf{J} - y\mathbf{K}$, so, along Γ, in terms of t,$\nabla w = 2e^t\mathbf{I} - e^{2t}\mathbf{J} - e^{t+1}\mathbf{K}$. Now

$$\frac{d\mathbf{X}}{dt} = e^t\mathbf{I} + e^{t+1}\mathbf{J} + 2e^{2t}\mathbf{K}\ ,$$

and

$$\frac{dw}{dt} = \nabla w \cdot \frac{d\mathbf{X}}{dt} = 2e^{2t} - e^{3t+1} - 2e^{3t+1} = 2e^{2t} - 3e^{3t+1}\ .$$

4. Let $f(x, y) = x^2y + 27y^2 + x$.

Solution. a) $\nabla f = (2xy + 1)\mathbf{I} + (x^2 + 54y)\mathbf{J}$

b) What are the critical points of f?

Solution. We must solve the pair of equations $2xy+1=0,\ x^2+54y=0$. The first gives $y=-1/2x$; substituting in the second gives $x^3=27$, so $x=3$ and $y=-1/6$. The only critical point is (3,-12/6).

c) What kind of critical points are they?

Solution. We calculate $f_{xx}=-1/3,\ f_{xy}=6,\ f_{yy}=54$, so $f_{xx}f_{yy}-f_{xy}^2=-54<0$, and the point is a saddle point.

5. Find the minimum value of $3x^2+y^2$ on the curve $x^2+xy=1$.

Solution. We use the method of Lagrange multipliers. Let $f(x,y)=3x^2+y^2, g(x,y)=x^2+xy$. We have

$$\nabla f=6x\mathbf{I}+2y\mathbf{J}\ ,\quad \nabla g=(2x+y)\mathbf{I}+x\mathbf{J}\ .$$

The equations to solve are

$$3x=\lambda(2x+y)\ ,\quad y=\lambda x\ ,\quad x^2+xy=1\ .$$

Substitute the second equation in the first to get $3x=\lambda(2x+\lambda x)$. Since $x=0$ does not solve the last equation, we can cancel x to get $3=2\lambda+\lambda^2$, which has the solutions $\lambda=1,-3$. The case $\lambda=1$ gives us $y=x$; substitute that in the last equation to get $2x^2=1$, or $x=y=\pm 1/\sqrt{2}$, and $f(x,y)=2$. The case $\lambda=-3$ leads to $y=-3x$; putting that in the last equation gives $-x^2=1$, an impossibility. Thus the only possible solution is at the first point. Since f is always positive, it must have a minimum, so since 2 is the only candidate, it is the minimum.