Calculus III
Exam 1, Summer 2003, Answers

1. $\mathbf{V} = 3\mathbf{I} - \mathbf{J}$, $\mathbf{W} = 2\mathbf{I} + 5\mathbf{J}$ are two vectors in the plane.
a) Find the angle between $\mathbf{V}$ and $\mathbf{W}$.

Answer. Let α be the angle. Then

$$\cos\alpha = \frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{V}||\mathbf{W}|} = \frac{3(2)-5}{\sqrt{10}\sqrt{29}} = \frac{1}{\sqrt{290}} = .0587\ ,$$

so $\alpha = 1.512$ radians, or $86.63°$.

b) Find the vector which is orthogonal to $\mathbf{V}$ and counterclockwise from $\mathbf{W}$.

Answer. Since $\mathbf{W}$ is counterclockwise from $\mathbf{V}$, and the angle α is less than a right angle, the vector we seek is $\mathbf{V}^{\perp} = \mathbf{I} + \mathbf{3J}$.

c) Find the area of the parallelogram spanned by $\mathbf{V}$ and $\mathbf{W}$.

Answer. The area is $\mathbf{V}^{\perp}\cdot\mathbf{W} = 2 + 15 = 17$.

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2. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = \ln t\,\mathbf{I} + \frac{1}{t}\mathbf{J}$$

Find the velocity, speed, acceleration, tangent and normal vectors, and normal acceleration of the particle at any time t.

Answer. Differentiate $\mathbf{X}(t)$:

$$\mathbf{V} = \frac{1}{t}\mathbf{I} - \frac{1}{t^2}\mathbf{J} = \frac{1}{t^2}(t\mathbf{I} - \mathbf{J})\ .$$

$$\mathbf{A} = -\frac{1}{t^2}\mathbf{I} + \frac{2}{t^3}\mathbf{J}\ .$$

$$\frac{ds}{dt} = \frac{1}{t^2}\sqrt{1+t^2}, \quad \mathbf{T} = \frac{t\mathbf{I} - \mathbf{J}}{\sqrt{1+t^2}}, \quad \mathbf{N} = \frac{\mathbf{I} + t\mathbf{J}}{\sqrt{1+t^2}}\ .$$

$$a_N = \mathbf{A}\cdot\mathbf{N} = \frac{1}{\sqrt{1+t^2}}\left(-\frac{1}{t^2} + \frac{2}{t^2}\right) = \frac{1}{t^2\sqrt{1+t^2}}\ .$$

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3. Find the equation of the plane through the point (0,-1,3) which is parallel to the vectors $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$ and $3\mathbf{I} - 2\mathbf{J} + \mathbf{K}$.

Answer. The normal to the plane is the cross product of the two given vectors. This is

$$\mathbf{N} = \det\begin{pmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 1 & -2 & 2 \\ 3 & -2 & 1 \end{pmatrix} = 2\mathbf{I} + 5\mathbf{J} + 4\mathbf{K}\ .$$

Taking $\mathbf{X_0} = (0)\mathbf{I} - \mathbf{J} + 3\mathbf{K}$, the equation is $(\mathbf{X} - \mathbf{X_0})\cdot\mathbf{N} = 0$, which comes to $2x + 5y + 4z = 7$.

4. Find the distance of the point $(2,0,1)$ from the line whose symmetric equations are

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-1}{-2}$$

Answer. Let $\mathbf{Q}$ be the given point. $\mathbf{P} = 2\mathbf{I} - \mathbf{J} + \mathbf{K}$ is a point on the line, and $\mathbf{L} = 3\mathbf{I} + 4\mathbf{J} - 2\mathbf{K}$ is a vector in the direction of the line. Then $\mathbf{Q} - \mathbf{P} = -\mathbf{J}$, and the desired distance is

$$\frac{|-\mathbf{J} \times \mathbf{L}|}{|\mathbf{L}|} = \sqrt{\frac{13}{29}} \, .$$

5. A particle moves in space according to the formula

$$\mathbf{X}(t) = e^t \mathbf{I} + e^{2t} \mathbf{J} - t\mathbf{K} \, .$$

Find the normal acceleration at the point $t = 0$.

Answer. Differentiate;

$$\mathbf{V}(t) = e^t \mathbf{I} + 2e^{2t} \mathbf{J} - \mathbf{K} \, , \quad \mathbf{A}(t) = e^t \mathbf{I} + 4e^{2t} \mathbf{J} \, .$$

Now, evaluate at $t = 0$ (don't work with the general formulas!) to find

$$\mathbf{V} = \mathbf{I} + 2\mathbf{J} - \mathbf{K}, \; \mathbf{A} = \mathbf{I} + 4\mathbf{J} \, .$$

Then $|\mathbf{V}| = \sqrt{6}$ and since $\mathbf{T} = \mathbf{V}/|\mathbf{V}|$ we have

$$\mathbf{A} \cdot \mathbf{T} = \frac{9}{\sqrt{6}} \, ,$$

$$\mathbf{A} - (\mathbf{A} \cdot \mathbf{T})\mathbf{T} = -\frac{1}{2}\mathbf{I} + \mathbf{J} + \frac{3}{2}\mathbf{K} \, ,$$

so $a_N = |\mathbf{A} - (\mathbf{A} \cdot \mathbf{T})\mathbf{T}| = \sqrt{7/2}$.