Calculus III, Mathematics 2210-90

Examination 1, Feb 5,7, 2004: Answers

1. $\mathbf{V} = 3\mathbf{I} - \mathbf{J}$, $\mathbf{W} = 2\mathbf{I} + 5\mathbf{J}$ are two vectors in the plane.
a) Find the unit vector $\mathbf{X}$ which is orthogonal to $\mathbf{V}$ so that $\mathbf{W}$ lies between $\mathbf{V}$ and $\mathbf{X}$.

Solution. Draw a diagram to see that $\mathbf{W}$ is counterclockwise from $\mathbf{V}$. Thus $\mathbf{V}^\perp = \mathbf{I} + 3\mathbf{J}$ gives the direction we want, and $\mathbf{X}$ is the unit vector in that direction:

$$\mathbf{X} = \frac{\mathbf{V}^\perp}{||\mathbf{V}^\perp||} = \frac{\mathbf{I} + 3\mathbf{J}}{\sqrt{10}} \ .$$

b) Find the area of the parallelogram spanned by $\mathbf{V}$ and $\mathbf{W}$.

Solution.

$$Area = \det \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix} = 15 - (-2) = 17 \ .$$

2. A particle moves in the plane according to the equation

$$\mathbf{X}(t) = t^3\mathbf{I} + (3t^2 - t)\mathbf{J}$$

Find the velocity, speed, acceleration, tangent and normal vectors, and normal acceleration of the particle at time $t = 2$.

Solution. First we differentiate:

$$\mathbf{V} = 3t^2\mathbf{I} + (6t - 1)\mathbf{J} \ , \qquad \mathbf{A} = 6t\mathbf{I} + 6\mathbf{J} \ .$$

Now, evaluate at $t = 2$ (because arithmetic is easier than algebra!):

$$\mathbf{V} = 12\mathbf{I} + 11\mathbf{J} \ , \qquad \mathbf{A} = 12\mathbf{I} + 6\mathbf{J} \ ,$$

$$\mathbf{T} = \frac{12\mathbf{I} + 11\mathbf{J}}{\sqrt{265}} \ , \qquad \mathbf{T}^\perp = \left(\frac{-11\mathbf{I} + 12\mathbf{J}}{\sqrt{265}}\right) \ ,$$

$$a_N = |\mathbf{A} \cdot \mathbf{T}^\perp| = \left|\frac{\mathbf{-132 + 72}}{\sqrt{\mathbf{265}}}\right| = \frac{60}{\sqrt{265}}$$

and

$$\mathbf{N} = \left(\frac{11\mathbf{I} - 12\mathbf{J}}{\sqrt{265}}\right) \ .$$

Remember that $\mathbf{N}$ is that unit vector orthogonal to $\mathbf{T}$ which has a positive dot product with $\mathbf{A}$.

3. Find the symmetric equations of the line through the point (0,-1,3) which is perpendicular to the vectors $\mathbf{I} - 2\mathbf{J} + 2\mathbf{K}$ and $3\mathbf{I} - 2\mathbf{J} + \mathbf{K}$.

Solution. The vector $\mathbf{L}$ in the direction of the line is orthogonal to both the given vectors, so is in the direction of their cross product. We calculate:

$$\mathbf{L} = \det \begin{pmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 1 & -2 & 2 \\ 3 & -2 & 1 \end{pmatrix} = 2\mathbf{I} + 5\mathbf{J} + 4\mathbf{K} \ .$$

Thus, since (0,-1,3) is on the line, we can take the symmetric equations as:

$$\frac{x}{2} = \frac{y+1}{5} = \frac{z-3}{4} \ .$$

4. Find the distance of the point $P(2,0,1)$ from the plane with equation $2x - 3y + z = 1$.

Solution. We find a point Q on the plane by setting x and y to zero and solving for z: $z = 1$. Thus $Q : (0,0,1)$ is on the plane. The distance then, is the length of the projection of the vector $PQ = 2\mathbf{I}$ in the direction normal to the plane; this is the direction of $\mathbf{N} = 2\mathbf{I} - 3\mathbf{J} + \mathbf{K}$. We calculate $PQ \cdot \mathbf{N} = 4$ and $|\mathbf{N}| = \sqrt{14}$, so the distance is $PQ \cdot \mathbf{N}/|\mathbf{N}| = 4/\sqrt{14}$.

5. A particle moves in space according to the formula $\mathbf{X}(t) = \ln t\mathbf{I} + t^2\mathbf{J} - t\mathbf{K}$.

a) Find the values of t at which the velocity and acceleration vectors are orthogonal. bi
Solution. First, differentiate to find the velocity and acceleration vectors:

$$\mathbf{V} = \frac{1}{t}\mathbf{I} + 2t\mathbf{J} - \mathbf{K} \qquad \mathbf{A} = -\frac{1}{t^2}\mathbf{I} + 2\mathbf{J} \ .$$

We are asked to find the values of t for which these vectors are orthogonal, so we solve the equation $\mathbf{V} \cdot \mathbf{A} = 0$:

$$-\frac{1}{t^3} + 4t = 0 \quad \text{or} \quad 4t^4 = 1 \ ,$$

which has the roots $t = \pm\sqrt{2}/2$.

b) What is the tangential acceleration of the particle at these points?

Solution. Since a_T is the length of the projection of $\mathbf{A}$ in the direction of $\mathbf{V}$, and at these points $\mathbf{A}$ is orthogonal to $\mathbf{V}$, we have $a_T = 0$.