Calculus III, Mathematics 2210-90

Examination 1, September 18,20, 2003, Answers

You may use graphing calculators and a Table of Integrals. Each problem is worth 20 points. You MUST show your work. Just the correct answer is not sufficient for any points.

1. On the plane, find the distance of the point (2,3) from the line whose equation is $6x - y = 2$.

Solution. Let P be the point (2,3) and let Q be any point on the line; we'll take $Q(1,4)$. The equation of the line can be written as $\mathbf{X} \cdot \mathbf{N} = 2$, where $\mathbf{N} = 6\mathbf{I} - \mathbf{J}$ is normal to the line. The distance we want is the length of the projection of $\overrightarrow{PQ} = -\mathbf{I} + \mathbf{J}$ in the direction of $\mathbf{N}$:

$$d = |proj_{\mathbf{N}}\overrightarrow{PQ}| = \frac{|\overrightarrow{PQ} \cdot \mathbf{N}|}{|\mathbf{N}|} = \frac{|-6-1|}{\sqrt{6^2+1}} = \frac{7}{\sqrt{37}} = 1.1508 \ .$$

Alternatively, we can use 3-D methods. The vector $\mathbf{L} = \mathbf{I} + 6\mathbf{J}$ lies in the direction of the line, and we have

$$|\overrightarrow{PQ} \times \mathbf{L}| = |\overrightarrow{PQ}||\mathbf{L}| \sin\theta = d|\mathbf{L}| \ .$$

Draw a picture to see this! So, we calculate $\overrightarrow{PQ} \times \mathbf{L} = -7\mathbf{K}$, so

$$d = \frac{|\overrightarrow{PQ} \times \mathbf{L}|}{|\mathbf{L}|} = \frac{7}{\sqrt{37}} \ .$$

2. In three dimensions, find the equation of the plane through the point $P(2, -1, 3)$ and perpendicular to the line segment from P to $Q(7, 2, 0)$.

Solution. The vector from Q tp P is $5\mathbf{I} + 3\mathbf{J} - 3\mathbf{K}$. This is the normal $\mathbf{N}$ to the plane. Thus, the equation of the plane is $(\mathbf{X} - \mathbf{P}) \cdot \mathbf{N} = 0$, which is, in coordinates

$$5(x-2) + 3(y+1) - 3(z-3) = 0 \quad \text{or} \quad 5x + 3y - 3z + 2 = 0 \ .$$

3. Find a vector $\mathbf{X}$ orthogonal to the vectors $\mathbf{V} = 2\mathbf{I} + \mathbf{K}, \ \mathbf{W} = \mathbf{I} - 2\mathbf{J} + 3\mathbf{K}$.

Solution. That is the cross product of $\mathbf{V}$ and $\mathbf{W}$:

$$\mathbf{V} \times \mathbf{W} = \det \begin{pmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2 & 0 & 1 \\ 1 & -2 & 3 \end{pmatrix} = 2\mathbf{I} - 5\mathbf{J} - 4\mathbf{K} \ .$$

4. A curve in the plane is given parametrically by the equations

$$x = t^2 - 1 \ , \qquad y = t^2 - t \ .$$

Find the curvature of the curve at the point (3,2), corresponding to $t = 2$.

Solution. Think of this as a the equations for a particle in motion, with t being time. Then we write this in vectors:

$$\mathbf{X} = (t^2 - 1)\mathbf{I} + (t^2 - t)\mathbf{J} \ ,$$

$$\mathbf{V} = 2t\mathbf{I} + (2t - 1)\mathbf{J} \ , \quad \mathbf{A} = 2\mathbf{I} + 2\mathbf{J} \ .$$

At $t = 2$, we have

$$\mathbf{V} = 4\mathbf{I} + 3\mathbf{J}, \ \mathbf{A} = 2\mathbf{I} + 2\mathbf{J} \ .$$

Thus $ds/dt = |\mathbf{V}| = 5$, and $\mathbf{T}^\perp = \pm(0.6\mathbf{I} - 0.8\mathbf{J})$.

$$a_N = \mathbf{A} \cdot \mathbf{T}^\perp = |2(0.6) - 2(0.8)| = 0.4$$

and the curvature is

$$\kappa = \frac{a_N}{(ds/dt)^2} = \frac{0.4}{25} = 0.016 \ .$$

5. A particle moves in space according to the formula

$$\mathbf{X}(t) = t^2\mathbf{I} + \ln t\mathbf{J} + (t^3 + t + 1)\mathbf{K} \ .$$

Find the tangential and normal components of its acceleration at the point $t = 1$.

Solution. Differentiating,

$$\mathbf{V} = 2t\mathbf{I} + \frac{1}{t}\mathbf{J} + (3t^2 + 1)\mathbf{K} \ , \quad \mathbf{A} = 2\mathbf{I} - \frac{1}{t^2}\mathbf{J} + 6t\mathbf{K} \ .$$

At $t = 1$, $\mathbf{V} = 2\mathbf{I} + \mathbf{J} + 4\mathbf{K}, \mathbf{A} = 2\mathbf{I} - \mathbf{J} + 6\mathbf{K}$.

$$a_T = \mathbf{A} \cdot \mathbf{T} = \frac{\mathbf{A} \cdot \mathbf{V}}{|\mathbf{V}|} = \frac{4 - 1 + 24}{\sqrt{4 + 1 + 16}} = \frac{27}{\sqrt{21}} = 5.892 \ ,$$

$$a_N\mathbf{N} = \mathbf{A} - a_T\mathbf{T} = 2\mathbf{I} - \mathbf{J} + 6\mathbf{K} - \frac{27}{21}(2\mathbf{I} + \mathbf{J} + 4\mathbf{K}) = \frac{-12\mathbf{I} - 48\mathbf{J} + 18\mathbf{K}}{21} \ .$$

a_N is the magnitude of this vector which is

$$a_N = \frac{6}{21}| - 2\mathbf{I} - 8\mathbf{J} + 3\mathbf{K}| = 2\sqrt{\frac{11}{7}} \ .$$

Alternatively,

$$a_N = \frac{|\mathbf{A} \times \mathbf{V}|}{|\mathbf{V}|} = \frac{|10\mathbf{I} - 4\mathbf{J} - 4\mathbf{K}|}{\sqrt{21}} = \sqrt{\frac{44}{7}} = 2\sqrt{\frac{11}{7}} \ .$$