Calculus III 2210-90 Exam 2

Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all your work in the space provided and circle your final answers.

1. (14pts) For this problem, consider the function

$$f(x, y) = x^2 + \sin(xy).$$

(a) (4pts) Find the gradient of f at $(1, 0)$, $\nabla f(1, 0)$.

4

$$\nabla f(x,y) = \langle 2x + y\cos(xy), x\cos(xy) \rangle$$

$$\nabla f(1,0) = \langle 2, 1 \rangle$$

(b) (3pts) Find the maximum rate of change of f at the point $(1, 0)$.

3

$$= \|\nabla f(1,0)\| = \sqrt{2^2+1^2} = \sqrt{5}$$

(c) (3pts) Let $\mathbf{u} = \langle -\frac{8}{17}, \frac{15}{17} \rangle$. Find $D_{\mathbf{u}} f(1, 0)$, that is, find the directional derivative of f at $(1, 0)$ in the direction of $\mathbf{u}$.

3

$$D_{\bar{u}} f(1,0) = \nabla f(1,0) \cdot \bar{u} = \langle 2,1 \rangle \cdot \langle -\tfrac{8}{17}, \tfrac{15}{17} \rangle = -\tfrac{16}{17} + \tfrac{15}{17} = -\tfrac{1}{17}$$

(d) (4pts) Find the equation of the tangent plane to the surface $z = x^2 + \sin(xy)$ at the point $(1, 0, 1)$.

4

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \quad 2$$

$$z = 1 + 2(x-1) + 1(y-0) = 2x + y - 1$$

2. (10pts) Ohm's Law states that the current I through a simple circuit is equal to the voltage V over the resistance R, or $I = \frac{V}{R}$. Therefore, if 5 volts are applied to a circuit that has 1 ohm of resistance, 5 amps of current will flow. Use differentials to estimate how the current changes if the voltage increases from 5 to 5.1 volts and the resistance decreases from 1 to .8 ohms.

$$I_V = \frac{1}{R}, \quad I_R = \frac{-V}{R^2}$$

10

$$dI = I_V\, dV + I_R\, dR$$

when $V = 5, R = 1$, $I_V = \frac{1}{1} = 1$ and $I_R = \frac{-5}{1^2} = -5$.

$dV = .1$ and $dR = -.2$

$$dI = (1)(.1) + (-5)(-.2) = 1.1 \text{ amps}$$

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3. (8pts) Consider the function

$$f(x,y) = x^2e^{-y} + ye^{-y} + 9.$$

(a) (4pts) Find the critical point(s) of f.

4

$f_x = 2xe^{-y} = 0 \Rightarrow x = 0.$

$f_y = -x^2e^{-y} + e^{-y} - ye^{-y} \overset{(x=0)}{=} e^{-y} - ye^{-y} = (1-y)e^{-y} = 0 \Rightarrow y = 1.$

So $(0,1)$ is only cp.

(b) (4pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$, and use it to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

4

$f_{xx} = 2e^{-y} \rightarrow f_{xx}(0,1) = 2/e$

$f_{yy} = x^2e^{-y} - 2e^{-y} + ye^{-y} \rightarrow f_{yy}(0,1) = -1/e.$

$f_{xy} = -2xe^{-y} \Rightarrow f_{xy}(0,1) = 0.$ $D(0,1) = (2/e)(-1/e) - 0^2 = -2/e^2 < 0.$

So $(0,1)$ is a saddle pt.

4. (10pts) Use Lagrange multipliers to find the maximum and minimum **values** of the function $f(x,y) = x + 8y$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

10

$\nabla f(x,y) = \langle 1, 8 \rangle$

$g(x,y) = \frac{x^2}{4} + y^2 - 1 = 0 \Rightarrow \nabla g(x,y) = \langle \frac{1}{2}x, 2y \rangle$

① $1 = \frac{\lambda}{2}x$

② $8 = 2\lambda y$

③ $\frac{x^2}{4} + y^2 = 1$

① $\Rightarrow \lambda = \frac{2}{x}.$

② $\Rightarrow \lambda = \frac{4}{y}$ $\Rightarrow \frac{2}{x} = \frac{4}{y} \Rightarrow 2y = 4x$ or $y = 2x.$

Now using ③

$1 = \frac{x^2}{4} + (2x)^2 = \frac{17}{4}x^2 \Rightarrow x^2 = \frac{4}{17} \Rightarrow x = \pm\frac{2}{\sqrt{17}}$

when $x = \frac{2}{\sqrt{17}}$, $y = \frac{4}{\sqrt{17}}$. when $x = \frac{-2}{\sqrt{17}}$, $y = \frac{-4}{\sqrt{17}}$.

$f(\frac{2}{\sqrt{17}}, \frac{4}{\sqrt{17}}) = \frac{2}{\sqrt{17}} + \frac{32}{\sqrt{17}} = \frac{34}{\sqrt{17}}$ max. $f(\frac{-2}{\sqrt{17}}, \frac{-4}{\sqrt{17}}) = -34/\sqrt{17}$ min

5. (8pts) The picture below is a contour plot of the function $f(x,y)$, along with four points labeled **A-D**. That is, the curves are level curves of the function $f(x,y)$ corresponding to the values written next to the curves. Answer the following questions by writing one letter **A-D** in each blank.

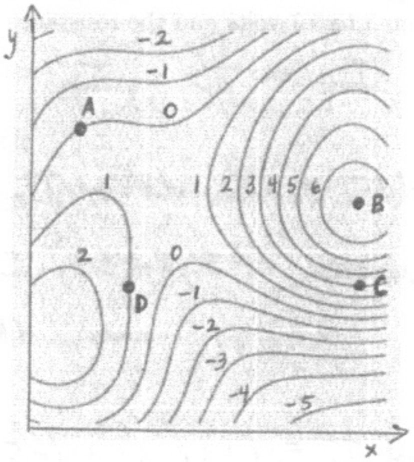
8

(a) __D__ At what point is the value of f equal to 1?

(b) __C__ At what point is ∇f parallel to **j**?

(c) __C__ At what point is $||\nabla f||$ the largest?

(d) __B__ At what point is ∇f equal to zero?



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6. (24pts) Evaluate the following integrals:

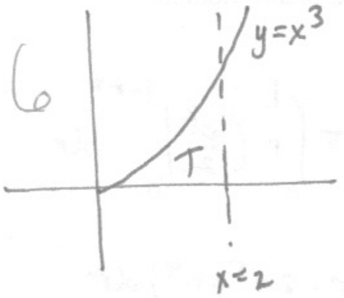
(a) (6pts) $\iint_R (x^2y+3)\,dA$, where R is the rectangle $R = \{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1\}$.

$$= \int_{-1}^{1}\int_{0}^{1}(x^2y+3)\,dy\,dx = \int_{-1}^{1}\left(\frac{x^2y^2}{2}+3y\right|_0^1 dx = \int_{-1}^{1}\left(\frac{x^2}{2}+3\right)dx$$

6

$$= \left(\frac{x^3}{6}+3x\right|_{-1}^{1} = \left(\tfrac{1}{6}+3\right)-\left(\tfrac{-1}{6}-3\right) = 6+\tfrac{1}{3} = \tfrac{19}{3}$$

(b) (6pts) $\iint_T xy\,dA$, where T is the region in the xy-plane bounded by the x-axis, the curve $y = x^3$, and the line $x = 2$.



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$$= \int_0^2\int_0^{x^3} xy\,dy\,dx = \int_0^2\left(\frac{xy^2}{2}\right|_0^{x^3} dx = \int_0^2 \frac{1}{2}x^7\,dx$$

$$= \left(\frac{1}{16}x^8\right|_0^2 = \frac{2^8}{16} = 16$$

(c) (6pts) $\iint_E \sqrt[4]{x^2+y^2}\,dA$, where E the annulus $4 \le x^2+y^2 \le 9$.

Use polar coordinates:

6

$$= \int_0^{2\pi}\int_2^3 (r^2)^{1/4}\,r\,dr\,d\theta = 2\pi\int_2^3 r^{3/2}\,dr = 2\pi\left(\frac{2}{5}r^{5/2}\right|_2^3$$

$$= \frac{4\pi}{5}\left(3^{5/2}-2^{5/2}\right)$$

(d) (6pts) $\iiint_B (xy+z\sin(y))\,dV$, where B is the box $B = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \pi, 0 \le z \le 2\}$.

$$= \int_0^1\int_0^\pi\int_0^2 (xy+z\sin(y))\,dz\,dy\,dx = \int_0^1\int_0^\pi \left(xyz+\frac{z^2}{2}\sin(y)\right|_0^2 dy\,dx$$

6

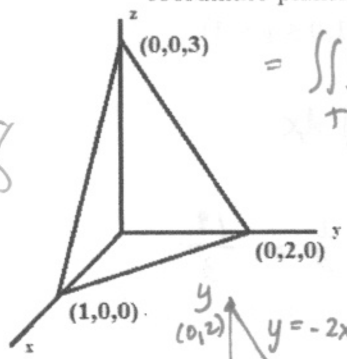
$$= \int_0^1\int_0^\pi (2xy+2\sin(y))\,dy\,dx = \int_0^1\left(xy^2-2\cos(y)\right|_0^\pi dx$$

$$= \int_0^1(\pi^2x+4) = \left(\frac{\pi^2x^2}{2}+4x\right|_0^1 = \frac{\pi^2}{2}+4$$

24

$$z = 3 - 3x - \frac{3}{2}y$$

7. (8pts) Use a triple integral to find the volume of the tetrahedron in the first octant determined by the coordinate planes and the plane $6x + 3y + 2z = 6$.



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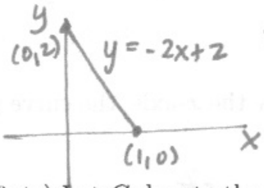
$$= \iiint_T 1\, dV = \int_0^1 \int_0^{-2x+2} \int_0^{3-3x-\frac{3}{2}y} 1 \, dz\, dy\, dx = \int_0^1 \int_0^{-2x+2} \left(3 - 3x - \frac{3}{2}y\right) dy\, dx$$

$$= \int_0^1 \left(3y - 3xy - \frac{3}{4}y^2 \Big|_0^{-2x+2}\right) dx = \int_0^1 \left(3(-2x+2) - 3x(-2x+2) - \frac{3}{4}(-2x+2)^2\right) dx$$

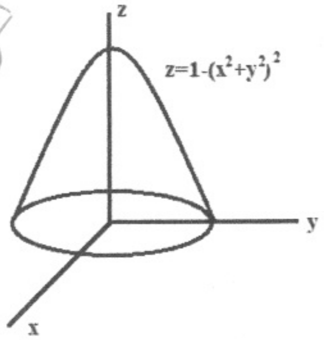


$$= \int_0^1 \left(-6x + 6 + 6x^2 - 6x - 3x^2 + 6x - 3\right) dx = \int_0^1 \left(3x^2 - 6x + 3\right) dx$$

$$= \left(x^3 - 3x^2 + 3x \Big|_0^1\right) = 1 - 3 + 3 = 1$$

8. (8pts) Let G denote the region bounded by the surface $z = 1 - (x^2 + y^2)^2$ and the xy-plane. Compute $\iiint_G z\, dV$.

Use cylindrical coords:



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$$\iiint_G z\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r^4} z r \, dz\, dr\, d\theta = 2\pi \int_0^1 r \left(\frac{z^2}{2}\Big|_0^{1-r^4}\right) dr$$

$$= 2\pi \int_0^1 \frac{r}{2}\left(1 - 2r^4 + r^8\right) dr = \pi \int_0^1 \left(r - 2r^5 + r^9\right) dr$$

$$= \pi\left(\frac{r^2}{2} - \frac{r^6}{3} + \frac{r^{10}}{10}\Big|_0^1\right) = \pi\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = \frac{8}{30}\pi$$

9. (10pts) In this problem, we will compute the improper integral $\iiint_{\mathbb{R}^3} e^{-(x^2+y^2+z^2)^{3/2}}\, dV$, where $\mathbb{R}^3$ denotes all of 3-space.

(a) (6pts) First compute $\iiint_{S_R} e^{-(x^2+y^2+z^2)^{3/2}}\, dV$, where S_R is the solid sphere of radius R, $S_R = \{(x, y, z) | x^2 + y^2 + z^2 \leq R^2\}$.

Use spherical coords:

6

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^R e^{-\rho^3} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = 2\pi \left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^R e^{-\rho^3} \rho^2 d\rho\right)$$

$u = -\rho^3$

$du = -3\rho^2 d\rho$

$$= 2\pi\left(-\cos\phi \Big|_0^{\pi}\right) \frac{-1}{3}\left(\int_0^{-R^3} e^u du\right) = 4\pi\left(\frac{-1}{3}\right)\left(e^{-R^3} - 1\right)$$

(b) (4pts) Now take the limit of your answer in part (a) as $R \to \infty$.

4

$$\lim_{R\to\infty} -\frac{4\pi}{3}\left(\overset{\to 0}{e^{-R^3}} - 1\right) = \frac{4\pi}{3}$$

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