1. (20pts) For this problem, consider the function
\[ f(x, y) = x^2 \cos y - 3xy. \]

(a) (3pts) Find the gradient of \( f \), \( \nabla f(x, y) \).

\[ \nabla f(x, y) = <2x \cos y - 3y, -x^2 \sin y - 3x> \]

(b) (3pts) Find the maximum rate of change of \( f \) at the point \((1, 0)\).

\[ \nabla f(1, 0) = <2, -3> \]
\[ \| \nabla f(1, 0) \| = \sqrt{2^2 + (-3)^2} = \sqrt{13} \]

(c) (3pts) Find unit vector which points in the direction in which the maximum rate of change occurs at \((1, 0)\).

\[ u = \frac{1}{\| \nabla f(1, 0) \|} \nabla f(1, 0) = \frac{1}{\sqrt{13}} <2, -3> = <\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}> \]

(d) (3pts) Let \( u = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \). Find \( D_u f(1, 0) \), that is, find the directional derivative of \( f \) at \((1, 0)\) in the direction of \( u \).

\[ (D_u f)(1, 0) = \nabla f(1, 0) \cdot u = <2, -3> \cdot <\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}> \]
\[ = \sqrt{2} - \frac{3\sqrt{2}}{2} = -\frac{\sqrt{2}}{2} \]

(e) (4pts) Find the equation of the tangent plane to the graph of \( f \) at \((1, 0)\).

\[ f(1, 0) = 1 \]
\[ z = 1 + 2(x-1) - 3y \]

(f) (4pts) Use part (e) above to estimate \( f \left( \frac{11}{10}, -\frac{1}{10} \right) \).

\[ f \left( \frac{11}{10}, -\frac{1}{10} \right) \approx 1 + 2 \left( \frac{11}{10} - 1 \right) - 3 \left( -\frac{1}{10} \right) \]
\[ = 1 + \frac{2}{10} + \frac{3}{10} = 1.5 \]
2. (13pts) Consider the function
\[ f(x, y) = x^2 + 2y^2 + x^2y - 1. \]

(a) (5pts) Find the three critical points of \( f \).
\[
\nabla f(x, y) = (2x + 2xy, 4y + x^2) \\
0 = 2x + 2xy = 2x(1+y) \Rightarrow x=0 \quad \text{or} \quad y=-1.
\]
When \( x=0 \), \( 0 = 4y \Rightarrow y = 0 \).
When \( y=-1 \), \( 0 = -4 + x^2 \Rightarrow x = \pm 2 \).

(b) (4pts) Find the discriminant, \( D = f_{xx}f_{yy} - (f_{xy})^2 \).
\[
f_{xx} = 2 + 2y \\
f_{yy} = 4 \\
f_{xy} = 2x \\
\text{\( D = (2+2y)(4) - 4x^2 \).}
\]

(c) (4pts) Use the discriminant to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.
\[
D(0,0) = 8 > 0 \Rightarrow (0,0) \text{ is a local min} \\
D(2,1) = -\frac{16}{25} < 0 \Rightarrow (2,1) \text{ is a saddle} \\
D(-2,-1) = -\frac{16}{25} < 0 \Rightarrow (-2,-1) \text{ is a saddle}
\]

3. (12pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function \( f(x, y) = x^2 + 2xy + y^2 \) on the circle \( x^2 + y^2 = 2 \).

\[ g(x, y) = x^2 + 2xy - 2 = 0. \]
\[ \nabla f = \lambda \nabla g \]
\[ g = 0 \]
\[ \begin{align*}
\langle 2x + 2y, 2x + 2y \rangle &= \lambda \langle 2x, 2y \rangle \\
x^2 + y^2 &= 2.
\end{align*} \]

\[ \lambda = 0 \]
\[ \text{\( x = y \) gives the pts:} \ (1,1) \text{ or } (-1,-1). \]
\[ \text{\( \lambda = 0 \) imps:} \]
\[ \text{\( x = -y \) gives the pts:} \ (1,1) \text{ or } (-1,-1). \]
\[ f(1,1) = f(-1,-1) = 4 \text{ max; } f(-1,1) = f(1,-1) = 0 \text{ min.} \]
4. (28pts) Find the following double integrals:

(a) (6pts) \( \int_0^1 \int_1^2 (x^3 - 3y^2x) \, dy \, dx \)
\[
= \int_0^1 \left( \int_1^2 (x^3 - 3y^2x) \, dy \right) \, dx \\
= \int_0^1 \left( x^3 - 3x^2 \right) \, dx \\
= \left[ \frac{1}{4}x^4 - \frac{3}{2}x^2 \right]_0^1 \\
= \frac{1}{4} - \frac{3}{2} = \frac{-13}{4}
\]

(b) (6pts) \( \iint_R (x^2 + y^2) \, dA \), where \( R \) is the rectangle \( R = \{(x,y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1\} \).
\[
= \int_{-1}^1 \int_{-1}^1 (x^2 + y^2) \, dx \, dy \\
= \int_{-1}^1 \left( \int_{-1}^1 (x^2 + y^2) \, dx \right) \, dy \\
= \int_{-1}^1 \left( \frac{1}{3} + y^2 + \frac{1}{3} + y^2 \right) \, dy \\
= \int_{-1}^1 \left( \frac{2}{3} + 2y^2 \right) \, dy \\
= \left[ \frac{2}{3}y + \frac{2}{5}y^3 \right]_{-1}^1 \\
= \frac{4}{3}
\]

(c) (8pts) \( \iint_T x \, dA \), where \( T \) is the region in the \( xy \)-plane bounded by the \( x \)-axis and the parabola \( y = -x^2 + 2x \).

\[
\iint_T x \, dA = \int_0^2 \int_0^{-x^2+2x} x \, dy \, dx \\
= \int_0^2 \left( \int_0^{-x^2+2x} x \, dy \right) \, dx \\
= \int_0^2 \left( -x^3 + 2x^2 \right) \, dx \\
= \left[ -\frac{1}{4}x^4 + \frac{2}{3}x^3 \right]_0^2 \\
= -4 + \frac{16}{3} = \frac{4}{3}
\]

(d) (8pts) \( \iint_D \cos(x^2 + y^2) \, dA \), where \( D \) the disk \( x^2 + y^2 \leq \frac{1}{2} \).

Use polar coordinates:
\[
\iint_D \cos(x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^{\sqrt{\frac{1}{2}}} \cos(r^2) \, r \, dr \, d\theta \\
= 2\pi \int_0^{\sqrt{\frac{1}{2}}} \cos(r^2) \, r \, dr \\
= \pi \int_0^{\pi/2} \cos(u) \, du \\
= \pi \left( \sin u \right)_0^{\pi/2} \\
= \pi
\]
5. (16pts) Evaluate the following triple integrals by using cylindrical or spherical coordinates:

(a) (8pts) \[ \iiint_S \sqrt{x^2 + y^2 + z^2} \, dV, \text{ where } S \text{ is sphere } x^2 + y^2 + z^2 \leq 2. \]

Use spherical coords:

\[ \iiint_S \sqrt{x^2 + y^2 + z^2} \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{\sqrt{2}} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \left( \int_0^{\pi} \sin \phi \, d\phi \right) \left( \int_0^{\sqrt{2}} \rho^2 \, d\rho \right) = 2\pi \left( \frac{1}{2} \right) \left( \frac{4}{3} \right) = \frac{4\pi}{3}. \]

(b) (8pts) \[ \iiint_B 2z \, dV, \text{ where } B \text{ is the piece of cylinder } x^2 + y^2 \leq 1 \text{ with } 0 \leq z \leq 1 + x^2 + y^2. \]

Use cylindrical coords:

\[ \iiint_B 2z \, dV = \int_0^{2\pi} \int_0^{1} \int_0^{1+r^2} 2r \, dz \, dr \, d\theta = 2\pi \int_0^{1} \left( r^2 + 1 \right) dr = 2\pi \left( \frac{1}{3} r^3 + r \right) \bigg|_0^1 = \frac{2\pi}{3}. \]

6. (11pts) Consider the map from the uv-plane to the xy-plane given by:

\[ x(u,v) = 2u + v \quad y(u,v) = u + 3v \]

This map takes the unit square S in the uv-plane to a parallelogram R in the xy-plane. See picture below.

(a) (4pts) Compute the Jacobian of this map. Recall that the Jacobian is the determinant of the matrix

\[ J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \]

\[ = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5. \]

(b) (7pts) Compute \( \iint_R (x + y) \, dA \) by using the change of variables formula

\[ \iint_R f(x, y) \, dx \, dy = \iint_S f(x(u,v), y(u,v)) |J(u,v)| \, du \, dv \]

\[ = \int_0^1 \int_0^1 \left[ (2u + v) + (u + 3v) \right] \cdot 5 \, du \, dv \]

\[ = 5 \int_0^1 \int_0^1 (3u + 4v) \, du \, dv \]

\[ = 5 \int_0^1 \left( \frac{3}{2} u^2 + 4uv \right) \bigg|_0^1 \, dv = 5 \int_0^1 (\frac{3}{2} + 4v) \, dv \]

\[ = 5 \left( \frac{3}{2} v + 2v^2 \right) \bigg|_0^1 = 5 \left( \frac{7}{2} \right) = \frac{35}{2}. \]