2210-90 Exam 2

Summer 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all your work in the space provided and circle your final answers.

1. (20pts) For this problem, consider the function

$$f(x,y) = x^2 \cos y - 3xy.$$

(a) (3pts) Find the gradient of f, $\nabla f(x,y)$.

3

$$\nabla f(x,y) = \langle 2x\cos y - 3y, -x^2 \sin y - 3x \rangle$$

(b) (3pts) Find the maximum rate of change of f at the point $(1,0)$.

3

$$\nabla f(1,0) = \langle 2, -3 \rangle$$

$$\|\nabla f(1,0)\| = \sqrt{2^2+3^2} = \sqrt{13}$$

(c) (3pts) Find unit vector which points in the direction in which the maximum rate of change occurs at $(1,0)$.

3

$$u = \frac{1}{\|\nabla f(1,0)\|} \nabla f(1,0) = \frac{1}{\sqrt{13}} \langle 2, -3 \rangle = \langle 2/\sqrt{13}, -3/\sqrt{13} \rangle$$

(d) (3pts) Let $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. Find $D_{\mathbf{u}} f(1,0)$, that is, find the directional derivative of f at $(1,0)$ in the direction of $\mathbf{u}$.

3

$$(D_u f)(1,0) = \nabla f(1,0) \cdot \bar{u} = \langle 2, -3 \rangle \cdot \langle \tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2} \rangle$$

$$= \sqrt{2} - \frac{3\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

(e) (4pts) Find the equation of the tangent plane to the graph of f at $(1,0)$.

4

$$f(1,0) = 1$$

$$z = 1 + 2(x-1) - 3y$$

(f) (4pts) Use part (e) above to estimate $f(\frac{11}{10}, -\frac{1}{10})$.

4

$$f\left(\tfrac{11}{10}, \tfrac{-1}{10}\right) \approx 1 + 2\left(\tfrac{11}{10} - 1\right) - 3\left(\tfrac{-1}{10}\right)$$

$$= 1 + \frac{2}{10} + \frac{3}{10} = 1.5$$

20

5

2. (13pts) Consider the function

$$f(x,y) = x^2 + 2y^2 + x^2y - 1.$$

(a) (5pts) Find the three critical points of f.

$\nabla f(x,y) = \langle 2x+2xy,\ 4y+x^2 \rangle$

$0 = 2x+2xy = 2x(1+y) \Rightarrow x=0$ or $y=-1$.

when $x=0$, $0=4y \Rightarrow y=0$.

when $y=-1$, $0=-4+x^2 \Rightarrow x=\pm 2$

$(0,0)$
$(2,-1)$
$(-2,-1)$

4

(b) (4pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$.

$f_{xx} = 2+2y$

$f_{yy} = 4$

$f_{xy} = 2x$

$D = (2+2y)\cdot 4 - 4x^2$

4

(c) (4pts) Use the discriminant to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

$D(0,0) = 8 > 0 \xRightarrow{f_{xx}>0} (0,0)$ is local min

$D(2,-1) = -16 < 0 \Rightarrow (2,-1)$ is a saddle

$D(-2,-1) = -16 < 0 \Rightarrow (-2,-1)$ is a saddle

12

3. (12pts) Use the method of Lagrange multipliers to find the maximum and minimum **values** of the function $f(x,y) = x^2 + 2xy + y^2$ on the circle $x^2 + y^2 = 2$.

$g(x,y) = x^2+y^2-2 = 0.$

$\nabla f = \lambda \nabla g$
$g = 0$

$\langle 2x+2y,\ 2x+2y \rangle = \lambda \langle 2x, 2y \rangle$
$x^2+y^2=2.$

① $2x+2y = 2\lambda x$
② $2x+2y = 2\lambda y$
③ $x^2+y^2 = 2$

① + ② imply
$2\lambda x = 2\lambda y$
which happens if
$x=y$ or $\lambda = 0$

$x=y$ gives the pts $(1,1)$ & $(-1,-1)$.

when $\lambda=0$, this implies $x=-y$ which gives the points $(1,-1)$ & $(-1,1)$.

$f(1,1) = f(-1,-1) = 4$ max

$f(-1,1) = f(1,-1) = 0$ min

25

4. (28pts) Find the following double integrals:

(a) (6pts) $\displaystyle\int_0^1 \int_1^2 (x^3 - 3y^2x)\ dy\ dx$

$$= \int_0^1 \left(x^3y - y^3x\right|_1^2 dx = \int_0^1 (2x^3 - 8x - x^3 + x)\,dx$$

$$= \int_0^1 (x^3 - 7x)\,dx = \left(\tfrac{1}{4}x^4 - \tfrac{7}{2}x^2\right|_0^1$$

$$= \frac{1}{4} - \frac{7}{2} = -\frac{13}{4}$$

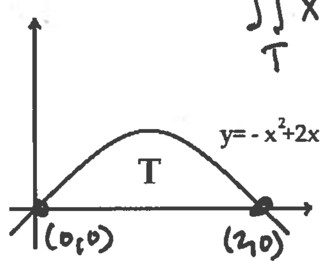
(b) (6pts) $\displaystyle\iint_R (x^2 + y^2)\ dA$, where R is the rectangle $R = \{(x, y) | -1 \le x \le 1, 0 \le y \le 1\}$.

$$= \int_0^1 \int_{-1}^1 (x^2+y^2)\,dx\,dy = \int_0^1 \left(\tfrac{1}{3}x^3 + y^2x\right|_{-1}^1 dy$$

$$= \int_0^1 \left(\tfrac{1}{3} + y^2 + \tfrac{1}{3} + y^2\right) dy = \int_0^1 \left(\tfrac{2}{3} + 2y^2\right) dy$$

$$= \left(\tfrac{2}{3}y + \tfrac{2}{3}y^3\right|_0^1 = \frac{4}{3}$$

(c) (8pts) $\displaystyle\iint_T x\ dA$, where T is the region in the xy-plane bounded by the x-axis and the parabola $y = -x^2 + 2x$.



$$\iint_T x\,dA = \int_0^2 \int_0^{-x^2+2x} x\ dy\,dx$$

$$= \int_0^2 (-x^3 + 2x^2)\,dx$$

$$= \left(-\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3\right|_0^2 = -4 + \frac{16}{3} = \frac{4}{3}$$

(d) (8pts) $\displaystyle\iint_D \cos(x^2 + y^2)\ dA$, where D the disk $x^2 + y^2 \le \frac{\pi}{2}$.

Use polar coordinates:

$$\iint_D \cos(x^2+y^2)\,dA = \int_0^{2\pi} \int_0^{\sqrt{\frac{\pi}{2}}} \cos(r^2)\,r\,dr\,d\theta$$

$$= 2\pi \int_0^{\sqrt{\pi/2}} \cos(r^2)\,r\,dr$$

$u = r^2$
$du = 2r\,dr$

$$= \pi \int_0^{\pi/2} \cos(u)\,du$$

$$= \pi \left(\sin u\right|_0^{\pi/2} = \pi$$

5. (16pts) Evaluate the following triple integrals by using cylindrical or spherical coordinates:

(a) (8pts) $\iiint_S \sqrt{x^2+y^2+z^2}\, dV$, where S is sphere $x^2+y^2+z^2 \le 2$.

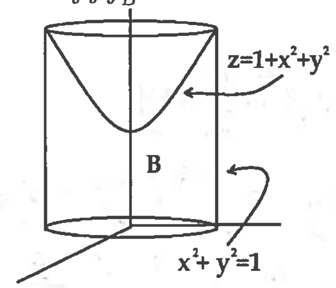
Use spherical coords:

$$\iiint_S \sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}} \rho^3 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \left(\int_0^{\pi} \sin\phi \, d\phi\right)\left(\int_0^{\sqrt{2}} \rho^3 \, d\rho\right)$$

$$= 2\pi (2)(1) = 4\pi$$

(b) (8pts) $\iiint_B 2z\, dV$, where B is the piece of cylinder $x^2+y^2 \le 1$ with $0 \le z \le 1+x^2+y^2$.



Use cylindrical coords:

$$\iiint_B 2z\,dV = \int_0^{2\pi}\int_0^{1}\int_0^{1+r^2} 2zr\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^1 \left(z^2 r\Big|_0^{1+r^2}\right) dr$$

$$= 2\pi \int_0^1 (r+2r^3+r^5)\,dr = 2\pi\left(\frac{r^2}{2}+\frac{1}{2}r^4+\frac{1}{6}r^6\Big|_0^1\right) = 2\pi\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{6}\right) = \frac{7\pi}{3}$$

6. (11pts) Consider the map from the uv-plane to the xy-plane given by

$$x(u,v) = 2u+v \qquad y(u,v) = u+3v$$

This map takes the unit square S in the uv-plane to a parallelogram R in the xy-plane. See picture below.

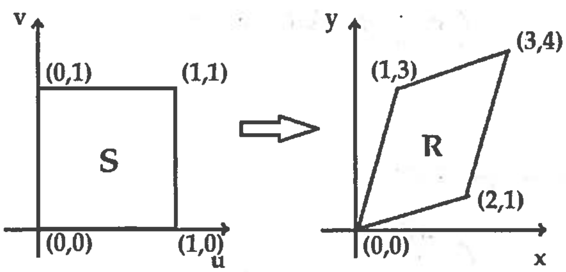
(a) (4pts) Compute the Jacobian of this map. Recall that the Jacobian is the determinant of the matrix

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$J(u,v) = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6-1 = 5.$$

(b) (7pts) Compute $\iint_R (x+y)\, dA$ by using the change of variables formula

$$\iint_R f(x,y)\, dx\, dy = \iint_S f(x(u,v), y(u,v))|J(u,v)|\, du\, dv$$



$$= \int_0^1\int_0^1 [(2u+v)+(u+3v)]\cdot 5\, du\, dv$$

$$= 5\int_0^1\int_0^1 (3u+4v)\,du\,dv$$

$$= 5\int_0^1 \left(\frac{3}{2}u^2+4uv\Big|_0^1\right) dv = 5\int_0^1 \left(\frac{3}{2}+4v\right) dv$$

$$= 5\left(\frac{3}{2}v+2v^2\Big|_0^1\right) = 5\left(\frac{7}{2}\right) = \frac{35}{2}$$