

2210-90 Exam 2
Spring 2014

Name

KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all your work in the space provided and circle your final answers.

1. (11pts) For this problem, consider the function

$$f(x, y) = x^2y^3 - ye^x + 9.$$

- (a) (4pts) Find the gradient of f at $(0, 2)$, $\nabla f(0, 2)$.

$$\nabla f(x, y) = \langle 2xy^3 - ye^x, 3x^2y^2 - e^x \rangle$$

$$\nabla f(0, 2) = \langle 0 - 2, -1 \rangle = \langle -2, -1 \rangle$$

- (b) (3pts) Find the maximum rate of change of f at the point $(0, 2)$.

$$\|\langle -2, -1 \rangle\| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}$$

- (c) (3pts) Let $\mathbf{u} = \langle \frac{5}{13}, -\frac{12}{13} \rangle$. Find $D_{\mathbf{u}}f(0, 2)$, that is, find the directional derivative of f at $(0, 2)$ in the direction of $\mathbf{u}$.

$$D_{\mathbf{u}}f(0, 2) = \nabla f(0, 2) \cdot \mathbf{u} = \langle -2, -1 \rangle \cdot \langle \frac{5}{13}, -\frac{12}{13} \rangle = \frac{12}{13} - \frac{10}{13} = \frac{2}{13}$$

2. (10pts) Consider the ellipsoid

$$3x^2 + y^2 + 2z^2 = 25.$$

- (a) Find a unit vector which is perpendicular to the tangent plane to the ellipsoid at the point $(1, 2, 3)$.

$$f(x, y, z) = 3x^2 + y^2 + 2z^2 = 25$$

$$\nabla f(x, y, z) = \langle 6x, 2y, 4z \rangle$$

$$\nabla f(1, 2, 3) = \langle 6, 4, 12 \rangle \text{ is perpendicular to tangent plane, but not a unit.}$$

$$\bar{\mathbf{n}} = \frac{1}{\sqrt{6^2 + 4^2 + 12^2}} \langle 6, 4, 12 \rangle = \frac{1}{14} \langle 6, 4, 12 \rangle = \langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \rangle$$

- (b) Find an equation for the tangent plane to the ellipsoid at $(1, 2, 3)$.

$$\bar{\mathbf{n}} = \langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7} \rangle \text{ and passes through } (1, 2, 3)$$

$$\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = d$$

$$\frac{3}{7}(1) + \frac{2}{7}(2) + \frac{6}{7}(3) = \frac{3}{7} + \frac{4}{7} + \frac{18}{7} = \frac{25}{7}$$

$$3x + 2y + 6z = 25$$

3. (8pts) Consider the function

$$f(x, y) = (x^2 + y^2)e^y.$$

(a) (4pts) Find the two critical points of f .

4

$$\nabla f(x, y) = \langle 2xe^y, 2ye^y + (x^2 + y^2)e^y \rangle = \langle 0, 0 \rangle$$

$$2xe^y = 0 \Rightarrow x = 0. \text{ When } x = 0,$$

$$0 = 2ye^y + (x^2 + y^2)e^y = (2y + y^2)e^y \Rightarrow 0 = 2y + y^2 \Rightarrow y = 0, -2.$$

cps: $(0, 0), (0, -2)$

(b) (4pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$, and use it to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

4

$$f_{xx} = 2e^y \quad D(0, 0) = (2)(2) - 0 = 4 > 0 \Rightarrow \text{local min.}$$

$$f_{xy} = 2xe^y \quad D(0, -2) = (2e^{-2})(-2e^{-2}) - 0 < 0 \Rightarrow \text{saddle.}$$

$$f_{yy} = (x^2 + y^2 + 2 + 4y)e^y$$

$$D = (2e^y)(x^2 + y^2 + 2 + 4y)e^y - 4x^2e^{2y}$$

4. (10pts) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2y^2$ on the circle $x^2 + y^2 = 1$. $= g(x, y)$

10

$$\nabla f(x, y) = \langle 2xy^2, 2x^2y \rangle$$

$$\nabla g(x, y) = \langle 2x, 2y \rangle$$

Now check values

$$f(0, 1) = f(0, -1) = f(1, 0) = f(-1, 0) = 0 \text{ min}$$

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = \frac{1}{4} \text{ no}$$

① $2xy^2 = 2\lambda x$

② $2x^2y = 2\lambda y$

③ $x^2 + y^2 = 1$

① $\Rightarrow 2xy^2 - 2\lambda x = 0$

$2x(y^2 - \lambda) = 0$

So $x = 0$ or $y^2 = \lambda$

• if $x = 0$, then ③ $\Rightarrow y = \pm 1$. (and $\lambda = 0$)

So we have pts $(0, 1)$ and $(0, -1)$

• if $y^2 = \lambda$ and $x \neq 0$, then ② $\Rightarrow x^2 = \lambda$.

Then ③ $\Rightarrow \lambda^2 + \lambda^2 = 1$ or $\lambda^2 = \frac{1}{2}$.

So we have the pts $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Finally, we have the possibility that $y = 0$ (from ②), which would give the pts $(1, 0)$ and $(-1, 0)$.

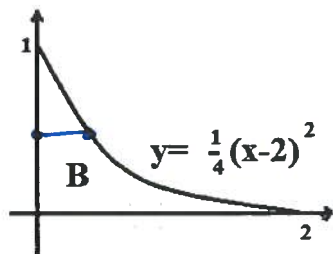
5. (6pts) The region B is bounded by the x -axis, the y -axis, and the curve $y = \frac{1}{4}(x-2)^2$. See the picture of B below. If I want to compute the integral of a function $f(x, y)$ over the region B , I can compute it in two equivalent ways:

6

$$\int_0^2 \int_0^{\frac{1}{4}(x-2)^2} f(x, y) dy dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy$$

where

$a = 0$ $b = 1$ $g_1(y) = 0$ $g_2(y) = 2\sqrt{y} + 2$



$$y = \frac{1}{4}(x-2)^2$$

$$4y = (x-2)^2$$

$$2\sqrt{y} = x-2$$

$$x = 2\sqrt{y} + 2$$

6. (28pts) Find the following double integrals:

(a) (6pts) $\int_0^{\pi/2} \int_0^2 (x + \sin(2y)) dx dy$

$$= \int_0^{\pi/2} \left(\frac{x^2}{2} + x \sin(2y) \right) \Big|_0^2 dy = \int_0^{\pi/2} (2 + 2 \sin(2y)) dy$$

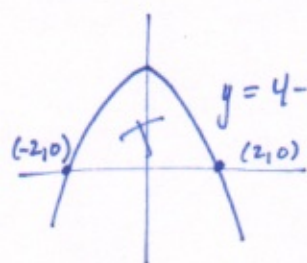
$$= (2y - \cos(2y)) \Big|_0^{\pi/2} = (\pi - \cos(\pi)) - (0 - \cos(0)) = \pi + 2$$

(b) (6pts) $\iint_R \frac{y}{x^2} dA$, where R is the rectangle $R = \{(x, y) | 1 \leq x \leq 2, 0 \leq y \leq 1\}$.

$$= \int_0^1 \int_1^2 y x^{-2} dx dy = \int_0^1 \left(-y x^{-1} \right) \Big|_1^2 dy = \int_0^1 \left(-\frac{y}{2} + y \right) dy = \int_0^1 \frac{y}{2} dy$$

$$= \left(\frac{y^2}{4} \right) \Big|_0^1 = \frac{1}{4}$$

(c) (8pts) $\iint_T x^2 y dA$, where T is the region in the xy -plane bounded by the x -axis and the curve $y = 4 - x^2$.



$$\iint_T x^2 y dA = \int_{-2}^2 \int_0^{4-x^2} x^2 y dy dx = \int_{-2}^2 \left(\frac{1}{2} x^2 y^2 \right) \Big|_0^{4-x^2} dx$$

$$= \int_{-2}^2 \frac{1}{2} x^2 (4-x^2)^2 dx = \frac{1}{2} \int_{-2}^2 x^2 (16 - 8x^2 + x^4) dx = \frac{1}{2} \int_{-2}^2 (16x^2 - 8x^4 + x^6) dx$$

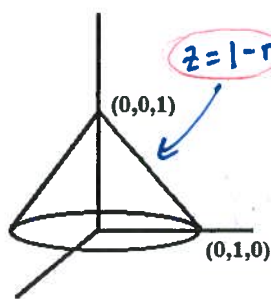
$$= \frac{1}{2} \left(\frac{16}{3} x^3 - \frac{8}{5} x^5 + \frac{1}{7} x^7 \right) \Big|_{-2}^2 = \frac{16}{3} (8) - \frac{8}{5} (32) + \frac{1}{7} (128) = \frac{1024}{105}$$

(d) (8pts) $\iint_D (x^2 + y^2)^{-1/2} dA$, where D the part of the disk $x^2 + y^2 \leq 16$ in the first quadrant.

Use polar coordinates: $D = \{(r, \theta) | 0 \leq r \leq 4, 0 \leq \theta \leq \frac{\pi}{2}\}$

$$= \int_0^{\pi/2} \int_0^4 \frac{1}{r} r dr d\theta = \int_0^{\pi/2} \int_0^4 dr d\theta = (4) \left(\frac{\pi}{2} \right) = 2\pi$$

7. (8pts) Use an integral in cylindrical coordinates to find the volume of the solid cone with radius of the base and height equal to 1. Recall that the volume of a region R can be computed as $\iiint_R 1 \, dV$.



$$\begin{aligned} \iiint_R 1 \, dV &= \int_0^{2\pi} \int_0^1 \int_0^{1-r} r \, dz \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r - r^2) \, dr \, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3} \right) \Big|_0^1 d\theta \\ &= 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{3} \end{aligned}$$

8. (8pts) Let S denote the spherical shell $S = \{(x, y, z) | 1 \leq x^2 + y^2 + z^2 \leq 4\}$. Compute

$$\begin{aligned} &\iiint_S \frac{1}{x^2 + y^2 + z^2} \, dV \\ &= \int_0^\pi \int_0^{2\pi} \int_1^2 \frac{1}{\rho^2} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \int_0^\pi \left(\int_0^{2\pi} \left(\int_1^2 d\rho \right) d\theta \right) \sin \phi \, d\phi \\ &= 2\pi \int_0^\pi \sin \phi \, d\phi = 2\pi \left(-\cos \phi \right) \Big|_0^\pi = 2\pi (-\cos(\pi) + \cos(0)) = 4\pi. \end{aligned}$$

9. (11pts) Consider the map from the uv -plane to the xy -plane given by

$$x(u, v) = u \quad y(u, v) = v + u^2$$

This map takes the unit square S in the uv -plane to the 'curvy' parallelogram R in the xy -plane. See picture below.

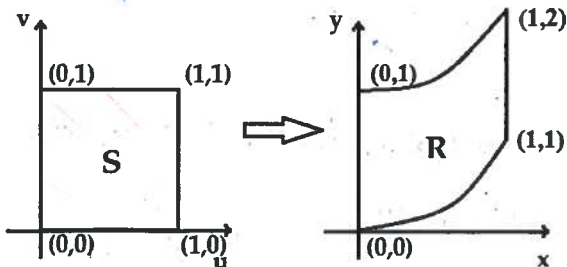
- (a) (4pts) Compute the Jacobian of this map. Recall that the Jacobian is the determinant $J(u, v) =$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = (1)(1) - (0)(2u) = 1$$

- (b) (7pts) Compute $\iint_R (2x + y) \, dA$ by using the change of variables formula

$$\iint_R f(x, y) \, dx \, dy = \iint_S f(x(u, v), y(u, v)) |J(u, v)| \, du \, dv$$



$$\begin{aligned} &= \int_0^1 \int_0^1 (2u + v + u^2) \, du \, dv \\ &= \int_0^1 \left(u^2 + vu + \frac{u^3}{3} \right) \Big|_0^1 dv = \int_0^1 \left(\frac{4}{3} + v \right) dv \\ &= \left(\frac{4}{3}v + \frac{v^2}{2} \right) \Big|_0^1 = \frac{4}{3} + \frac{1}{2} = \frac{11}{6} \end{aligned}$$