2210-90 Exam 2

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Page 5 is blank in case you need extra paper. Please circle your final answers.

1. (12pts, 4pts each) Evaluate the following limits. If the limit does not exist, write 'DNE' as your answer.

4 (a) $\lim_{(x,y)\to(1,0)} \frac{\cos(xy)}{1+x^2-y^2}$ $= \frac{\lim_{(x,y)\to(1,0)} \cos(xy)}{\lim_{(x,y)\to(1,0)} 1+x^2-y^2} = \frac{\cos(0)}{2} = \frac{1}{2}$

4 (b) $\lim_{(x,y)\to(0,0)} \frac{x-2y}{x+y}$

If we approach $(0,0)$ along x-axis $(y=0)$ then $\lim_{(x,y)\to(0,0)} \frac{x-2y}{x+y} = \lim_{x\to 0} \frac{x}{x} = 1$.

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If we approach $(0,0)$ along y-axis $(x=0)$ then $\lim_{(x,y)\to(0,0)} \frac{x-2y}{x+y} = \lim_{y\to 0} \frac{-2y}{y} = -2$. So limit DNE

4 (c) $\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2}$

Use polar coordinates

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = \lim_{r\to 0} \frac{r^4\sin^4\theta}{r^2} = \lim_{r\to 0} r^2\sin^4\theta = 0$$

independent of θ.

2. (12pts, 4pts each) Find the indicated partial derivatives of the function $f(x,y) = \sin(xy+y^2)$.

4 (a) $f_x(x,y) =$

$y\cos(xy+y^2)$

4 (b) $f_y(x,y) =$

$(x+2y)\cos(xy+y^2)$

4 (c) $f_{xy}(x,y) =$

$\cos(xy+y^2) - y(x+2y)\sin(xy+y^2)$

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3. (13pts) For this problem, consider the function

$$f(x, y) = xy^2 + x^3 - xy^3.$$

(a) (4pts) Find $\nabla f(1,1)$, that is, find the gradient of f at $(1,1)$.

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2 $\nabla f(x,y) = \langle y^2+3x^2-y^3,\ 2xy-3xy^2 \rangle$

2 $\nabla f(1,1) = \langle 3, -1 \rangle$

(b) (3pts) Find the maximum rate of change of f at $(1,1)$.

3 $\|\nabla f(1,1)\| = \sqrt{3^2+(-1)^2} = \sqrt{10}$

(c) (3pts) Let $\mathbf{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$. Find $D_{\mathbf{u}}f(1,1)$, that is, find the directional derivative of f at $(1,1)$ in the direction of $\mathbf{u}$.

3 $D_u f(1,1) = \bar{u} \cdot \nabla f(1,1) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle \cdot \langle 3, -1 \rangle = \sqrt{2}$

(d) (3pts) Find the unit vector that indicates the direction in which f increases the fastest at $(1,1)$.

3 $\bar{u} = \frac{1}{\|\nabla f(1,1)\|} \nabla f(1,1) = \frac{1}{\sqrt{10}} \langle 3, -1 \rangle = \langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$

4. (12pts) Find the equation for that tangent plane to the ~~ellipse~~ ellipsoid

$$3x^2 + y^2 + 2z^2 = 6$$

at the point $(1, -1, -1)$.

Use implicit differentiation

$\frac{\partial}{\partial x}\left(3x^2+y^2+2z^2=6\right)$

$6x + 4z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-3x}{2z} \Rightarrow \frac{\partial z}{\partial x}(1,-1,-1) = \frac{3}{2}$

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$\frac{\partial}{\partial y}\left(3x^2+y^2+2z^2=6\right)$

$2y + 4z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-y}{2z} \Rightarrow \frac{\partial z}{\partial x}(1,-1,-1) = -\frac{1}{2}$

$z = -1 + \frac{3}{2}(x-1) - \frac{1}{2}(y+1)$

Can also solve $z = -\sqrt{3-\frac{3}{2}x^2-\frac{1}{2}y^2}$ and use typical tangent plane computations

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5. (11pts) Consider the function

$$f(x, y) = x^3 + y^2 - 3x + 6y.$$

(a) (4pts) Find the 2 critical points of f.

4

2 $\nabla f(x,y) = \langle 3x^2-3, 2y+6 \rangle$

$3x^2 - 3 = 0 \Rightarrow x = \pm 1$ cps: $(1,-3)$ + $(-1,-3)$

$2y + 6 = 0 \Rightarrow y = -3$ 2

(b) (3pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$.

3

$f_{xx} = 6x$

$f_{yy} = 2$ $D = (6x)(2) - (0)^2 = 12x$

$f_{xy} = 0$

(c) (4pts) Use the discriminant to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

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2 $D(1,-3) = 12 > 0$ + $f_{xx}(1,-3) = 6 > 0 \Rightarrow (1,-3)$ is a local min

2 $D(-1,-3) = -12 < 0 \Rightarrow (-1,-3)$ is a saddle pt.

6. (12pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2x - 2y$ on the circle $x^2 + y^2 = 2$.

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2 $\nabla f(x,y) = \langle 2x-2, 2y-2 \rangle$

$g(x,y) = x^2 + y^2 - 2 = 0.$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

I get 3 equs:

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① $2x - 2 = 2x\lambda \Rightarrow x - 1 = x\lambda \Rightarrow x = \frac{1}{1-\lambda}$

② $2y - 2 = 2y\lambda \Rightarrow y - 1 = y\lambda \Rightarrow y = \frac{1}{1-\lambda}$ so $\underline{x = y}$

③ $x^2 + y^2 = 2$

Now use ③ knowing $x = y$.

③ $x^2 + x^2 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$ 2

This gives 2 points: $(1,1)$ and $(-1,-1)$.

$f(1,1) = -2$ min

$f(-1,-1) = 6$ max 2

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7. (28pts) Find the following double integrals:

(a) (6pts) $\int_1^3 \int_0^2 (xy - y^2)\, dx\, dy$

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$$= \int_1^3 \left(\frac{x^2}{2}y - xy^2 \Big|_0^2 \right) dy = \int_1^3 (2y - 2y^2)\, dy = \left(y^2 - \frac{2}{3}y^3 \Big|_1^3 \right)$$

$$= (9 - 18) - (1 - \tfrac{2}{3})$$

$$= -9\tfrac{1}{3} = -\frac{28}{3}$$

(b) (6pts) $\iint_R (x + 4y)\, dA$, where R is the rectangle $R = \{(x, y) | 0 \le x \le 2, 0 \le y \le 1\}$.

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$$= \int_0^1 \int_0^2 (x + 4y)\, dx\, dy = \int_0^1 \left(\frac{x^2}{2} + 4xy \Big|_0^2 \right) dy$$

$$= \int_0^1 (2 + 8y)\, dy = \left(2y + 4y^2 \Big|_0^1 \right) = 6$$

(c) (8pts) $\iint_T xy\, dA$, where T is the triangle with vertices $(0, 0)$, $(3, 0)$, and $(3, 6)$.

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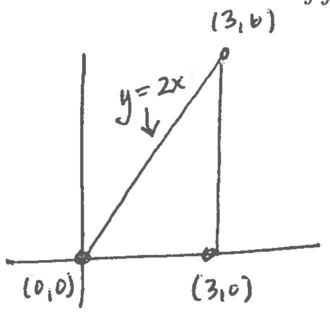

$$= \int_0^3 \int_0^{2x} xy\, dy\, dx = \int_0^3 \left(\frac{xy^2}{2} \Big|_0^{2x} \right) dx$$

$$= \int_0^3 2x^3\, dx = \left(\frac{x^4}{2} \Big|_0^3 \right) = \frac{81}{2}$$

or

$$\int_0^6 \int_{y/2}^3 xy\, dx\, dy = \frac{9}{4}(36) - \frac{6^4}{32}$$

(d) (8pts) $\iint_D \frac{xy}{x^2 + y^2}\, dA$, where D is the piece of the unit disk in the first quadrant, i.e. the set of points (x, y) where $x^2 + y^2 \le 1$ and both x and y are greater than or equal to zero.

Use polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

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$$\iint_D \frac{xy}{x^2+y^2}\, dA = \int_0^{\pi/2} \int_0^1 \frac{r^2 \cos\theta \sin\theta}{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r \cos\theta \sin\theta\, dr\, d\theta$$

$$= \left(\int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \right) \left(\int_0^1 r\, dr \right) = \left(\int_0^1 u\, du \right) \left(\frac{1}{2} \right) = \frac{1}{4}$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

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