2210-90 Exam 2

Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all your work in the space provided and circle your final answers.

1. (13pts) For this problem, consider the function

$$f(x, y) = (x^2 + y^2)e^{-x}.$$

(a) (4pts) Find the gradient of f, $\nabla f(x, y)$.

4

$$\nabla f(x,y) = \langle 2xe^{-x} - (x^2+y^2)e^{-x},\ 2ye^{-x} \rangle$$

(b) (3pts) Find the maximum rate of change of f at the point $(0, 1)$.

3

$$\nabla f(0,1) = \langle -1, 2 \rangle.$$

max rate of change $= \|\nabla f(0,1)\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

(c) (3pts) Find unit vector which points in the direction in which the maximum rate of change occurs at $(0, 1)$.

3

$$\bar{u} = \frac{1}{\|\nabla f(0,1)\|} \nabla f(0,1) = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

(d) (3pts) Let $\mathbf{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$. Find $D_{\mathbf{u}} f(0, 1)$, that is, find the directional derivative of f at $(0, 1)$ in the direction of $\mathbf{u}$.

3

$$D_{\bar{u}} f(0,1) = \nabla f(0,1) \cdot \bar{u} = \langle -1, 2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle = \frac{3}{5} + \frac{8}{5} = \frac{11}{5}$$

2. (8pts) Find the equation of the tangent plane to the surface at the point $(1, -1, -1)$.

$$x^2 + 3y^2 - 2z^2 = 2$$

8

Find $\partial z/\partial x$ and $\partial z/\partial y$ implicitly:

$$\frac{\partial}{\partial x}\left(x^2 + 3y^2 - 2z^2 = 2\right) \Rightarrow 2x - 4z\frac{\partial z}{\partial x} = 0 \xRightarrow{\text{at } (1,-1,-1)} 2 + 4\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-1}{2}$$

$$\frac{\partial}{\partial y}\left(x^2 + 3y^2 - 2z^2 = 2\right) \Rightarrow 6y - 4z\frac{\partial z}{\partial y} = 0 \xRightarrow{\text{at } (1,-1,-1)} -6 + 4\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{3}{2}$$

$$z = -1 - \frac{1}{2}(x-1) + \frac{3}{2}(y+1)$$

Another method:

$$f(x,y,z) = x^2 + 3y^2 - 2z^2 - 2 = 0$$

$$\nabla f = \langle 2x, 6y, -4z \rangle$$

$$\nabla f(1,-1,-1) = \langle 2, -6, 4 \rangle$$

$$0 = \langle 2, -6, 4 \rangle \cdot \langle x-1, y+1, z+1 \rangle$$

$$0 = 2(x-1) - 6(y+1) + 4(z+1)$$

3. (8pts) Consider the function
$$f(x,y) = x^3 + y^2 - 3x - 6y + 1.$$

(a) (4pts) Find the critical point(s) of f.

4

$\nabla f = \langle 3x^2-3,\ 2y-6 \rangle = \langle 0,0 \rangle.$

$3x^2-3=0 \Rightarrow x=\pm 1$

$2y-6=0 \Rightarrow y=3.$

So $(1,3)$ + $(-1,3)$ are cps.

(b) (4pts) Find the discriminant, $D = f_{xx}f_{yy} - (f_{xy})^2$, and use it to determine whether each of the critical points found in part (a) is a local minimum, a local maximum, or a saddle point.

4

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = 0.$

$D = (6x)(2) - 0^2 = 12x.$

$D(1,3) = 12 > 0$, since $f_{xx}(1,3) = 6 > 0 \Rightarrow (1,3)$ is local min

$D(-1,3) = -12 < 0 \Rightarrow (-1,3)$ is a saddle.

4. (10pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = x^2y$ on the circle $x^2+y^2=3$.

$g(x,y) = x^2+y^2-3=0.$

10

$\nabla f(x,y) = \lambda \nabla g(x,y)$, $g(x,y)=0$ $\Rightarrow$

$2xy = 2\lambda x$ ①

$x^2 = 2\lambda y$ ②

$x^2+y^2=3$ ③

① $\Rightarrow xy - \lambda x = x(y-\lambda) = 0$. So either $x=0$ or $y=\lambda$

- when $x=0$, ③ $\Rightarrow y = \pm\sqrt{3}$ and ② $\Rightarrow \lambda = 0.$
- when $y=\lambda$, ② $\Rightarrow x^2 = 2y^2 \overset{③}{\Rightarrow} 3y^2=3 \Rightarrow y^2=1 \Rightarrow y=\pm 1.$ Since $x^2=2y^2 \Rightarrow x^2=2$ or $x=\pm\sqrt{2}.$

So I check my fcn at $(0,\pm\sqrt{3}), (\sqrt{2},\pm 1), (-\sqrt{2},\pm 1)$ (then $\lambda = \pm 1$ appropriately).

$f(0,\sqrt{3}) = f(0,-\sqrt{3}) = 0.$

$f(+\sqrt{2},1) = f(-\sqrt{2},1) = 2$

$f(+\sqrt{2},-1) = f(-\sqrt{2},-1) = -2$

max value $=2$
min value $=-2$

5. (6pts) A factory manufactures aluminum cans. Ideally, the can produced has a height of 8 cm and a base radius of 2 cm. The volume of this ideal can is 32π cm^3 (since $V = \pi r^2 h$). However, due to errors in the manufacturing process, the cans produced actually measure $8 \pm .05$ cm high and $2 \pm .01$ cm in radius. Use differentials to estimate the maximum error in the volume of the can.

6

$V = \pi r^2 h$

$r=2$
$h=8$
$dr=.01$
$dh=.05$

$dV = \pi r^2\, dh + 2\pi r h\, dr$

$dV = \pi \cdot 4(.05) + 2\pi(2)(8)(.01)$

$= \pi(.2+.32) = .52\pi$ cm^3

24

6. (28pts) Find the following double integrals:

(a) (6pts) $\int_1^2 \int_0^{\pi} x \sin y \; dy \; dx$ $= \int_1^2 \left(-x\cos y \Big|_0^{\pi} dx = \int_1^2 2x \, dx\right.$

$= \left(x^2\Big|_1^2 = 4 - 1 = 3\right.$

6

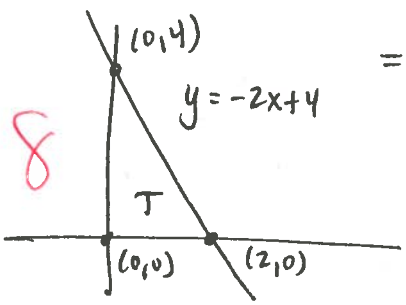
(b) (6pts) $\iint_R (x + 2x^2y) dA$, where R is the rectangle $R = \{(x, y)|0 \le x \le 2, -1 \le y \le 1\}$.

$= \int_0^2 \int_{-1}^1 (x + 2x^2y) \, dy \, dx = \int_0^2 \left(xy + x^2y^2\Big|_{-1}^1 dx\right. = \int_0^2 2x \, dx$

6

$= \left(x^2\Big|_0^2 = 4\right.$

(c) (8pts) $\iint_T (2y + 1) \; dA$, where T is the region in the xy-plane bounded by the x-axis, the y-axis, and the line $y = -2x + 4$.



8

$= \int_0^2 \int_0^{-2x+4} (2y+1) \, dy \, dx = \int_0^2 \left(y^2 + y\Big|_0^{-2x+4} dx\right.$

$= \int_0^2 \left((-2x+4)^2 + (-2x+4)\right) dx = \int_0^2 (4x^2 - 18x + 20) \, dx$

$= \left(\frac{4}{3}x^3 - 9x^2 + 20x\Big|_0^2 = \frac{32}{3} - 36 + 40 = \cancel{\frac{20}{3}}\right.$ $= \frac{44}{3}$

(d) (8pts) $\iint_D \frac{1}{(x^2 + y^2)} \; dA$, where D the annulus $1 \le x^2 + y^2 \le 4$.

Use polar coordinates

8 $= \int_0^{2\pi} \int_1^2 \frac{1}{r^2} \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^2 \frac{1}{r} \, dr \, d\theta$

$= \int_0^{2\pi} \left(\ln r\Big|_1^2 d\theta = \int_0^{2\pi} \ln 2 \, d\theta\right.$

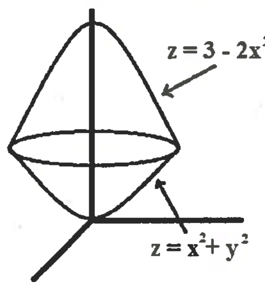
$= 2\pi \ln 2$

7. (6pts) The region B is bounded by the x-axis, the line $x = 4$, and the curve $y = \sqrt{x}$. See the picture of B at the bottom of the page. If I want to compute the integral of a function $f(x, y)$ over the region B, I can compute it in two equivalent ways:

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$$

where $a =$ <u>0</u> $b =$ <u>2</u> $g_1(y) =$ <u>y^2</u> $g_2(y) =$ <u>4</u>

8. (8pts) Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 3 - 2x^2 - 2y^2$. Recall that the volume of a region R can be computed as $\iiint_R 1\, dV$.



Use cylindrical coordinates 4

$$= \int_0^{2\pi} \int_0^1 \int_{r^2}^{3-2r^2} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_0^1 r(3 - 3r^2)\, dr = 2\pi \int_0^1 3r - 3r^3\, dr$$

$$= 2\pi \left(\frac{3}{2}r^2 - \frac{3}{4}r^4\right)\Big|_0^1 = 2\pi\left(\frac{3}{2} - \frac{3}{4}\right) = \frac{3}{2}\pi$$

9. (8pts) Let S denote the unit ball (solid sphere), $S = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$. Compute

$$\iiint_S \sqrt{x^2 + y^2 + z^2}\, dV$$

Switch to spherical

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^1 \rho^3\, d\rho\right)$$

$$= 2\pi \left(-\cos\phi \Big|_0^{\pi}\right)\left(\frac{1}{4}\rho^4 \Big|_0^1\right)$$

$$= 2\pi (2)\left(\frac{1}{4}\right) = \pi$$

10. (5pts) Consider the map from the uv-plane to the xy-plane given by

$$x(u, v) = u^3 + 5v \qquad y(u, v) = ue^v$$

Compute the Jacobian of this map. Recall that the Jacobian is the determinant of the matrix

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$J = \begin{vmatrix} 3u^2 & 5 \\ e^v & ue^v \end{vmatrix} = (3u^2)(ue^v) - (5)(e^v)$$

$$= 3u^3 e^v - 5e^v$$